Lecture 3

Bolza Problem

$$\inf_{\underline{\theta} \in L^\infty([t_0,t_1],\Theta)} J[\underline{\theta}] = \Phi(x(t_1)) + \int_{t_0}^{t_1} L(t, x(t), \theta(t))\, dt$$

$$\text{s.t.} \quad \dot{x}(t) = f(t, x(t), \theta(t)), \quad x(t_0) = x_0, \quad \theta(t) \in \Theta$$
$$t \in [t_0, t_1]$$

Alternatively, this is constrained optimization problem

$$\inf_{\underline{\theta}, \underline{x}} J[\underline{\theta}, \underline{x}] = \ldots$$
$$\text{subj to} \quad \dot{x}(t) = f(t, x(t), \theta(t))$$

- Lagrange multipliers.

$$\min_z \Phi(z) \text{ subj } g(z) = 0 \longrightarrow \Phi(z) + \lambda g(z) = \mathcal{L}(z,\lambda)$$
$$\partial_z \mathcal{L} = 0, \quad \partial_\lambda \mathcal{L} = 0$$

- $\Phi \equiv 0$ (Lagrange problem)

$$\inf_{\underline{\theta}, \underline{x}} \int_{t_0}^{t_1} L(t, x(t), \theta(t))\, dt \quad \text{subj to} \quad \dot{x}(t) - f(t, x(t), \theta(t)) = 0 .$$

$$\mathcal{L} = \int_{t_0}^{t_1} L(t, x(t), \theta(t))\, dt + \int_{t_0}^{t_1} \lambda(t) \left[\dot{x}(t) - f(t, x(t), \theta(t)) \right] dt .$$

$\nabla_{\lambda(t)}$: $\quad \dot{x} = f(t, x, \theta) \quad (\nabla_\lambda H)$

$\nabla_{x(t)}$:

$$\nabla_x L(t, x(t), \theta(t)) - \dot{\lambda}(t) - \nabla_x f(t, x(t), \theta(t))^T \lambda(t) = 0$$

$$\dot{\lambda}(t) = - \nabla_x \big(\underbrace{f(t, x(t), \theta(t))^T \lambda(t) - L(t, x(t), \theta(t))}_{H(t, x(t), \lambda(t), \theta(t))} \big)$$

$\nabla_{\theta(t)}$: $\quad \mathcal{L} = \int_{t_0}^{t_1} H(t, x(t), \lambda(t), \theta(t))\, dt + \int_{t_0}^{t_1} \lambda(t) \dot{x}(t)\, dt .$

$$\nabla_\theta H(t, x(t), \lambda(t), \theta(t)) = 0$$

Pontryagin's Maximum Principle

Define Hamiltonian $H : [t_0, t_1] \times \mathbb{R}^d \times \mathbb{R}^d \times \Theta \to \mathbb{R}$.

$$H(t, x, p, \theta) = p^T f(t, x, \theta) - L(t, x, \theta)$$

Theorem (PMP)

Let $\underline{\theta}^*$ be an optimal control, $\underline{x}^*$ be its controlled trajectory.
Then, there exists $p^* : [t_0, t_1] \to \mathbb{R}^d$ which is abs. cts. and

$$\begin{cases} \dot{x}^*(t) = f(t, x^*(t), \theta^*(t)) = \nabla_p H(t, x^*(t), p^*(t), \theta^*(t)), & x^*(0) = x_0 \\ \dot{p}^*(t) = -\nabla_x H(t, x^*(t), p^*(t), \theta^*(t)), & p^*(T) = -\nabla_x \Phi(x^*(t_1)) \\ H(t, x^*(t), p^*(t), \theta^*(t)) \geq H(t, x^*(t), p^*(t), \theta) & \text{for all } \theta \in \Theta \end{cases}$$

holds t- a.e.

p^* is called the co-state / adjoint.

Proof of the PMP

Step 1 : Conversion to Mayer Problem $(L \equiv 0)$

Define $\dot{x}^0(t) = L(t, x(t), \theta(t))$, $x^0(0) = 0$ $\Rightarrow$ $x^0(t) = \int_0^t L(\cdots)\,dt$.

$(x^0, x) \to \tilde{x}$, $(L, f) \to \tilde{f}$, $\tilde{\Phi}(\tilde{x}) = \Phi(x) + x^0$.

$\Rightarrow \inf_{\underline{\theta}} \Phi(x(t_1))$ subj $\dot{x} = f(t, x, \theta)$. $x(0) = x_0$.

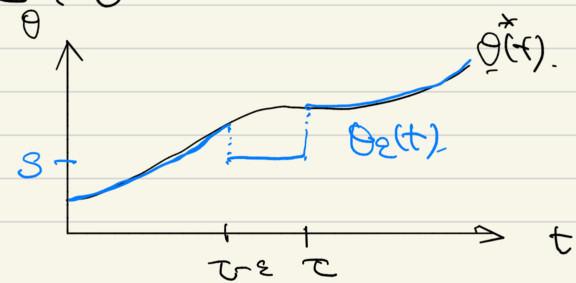
Step 2 : Needle perturbation.

Let $\underline{\theta}^*$ be an optimal control.

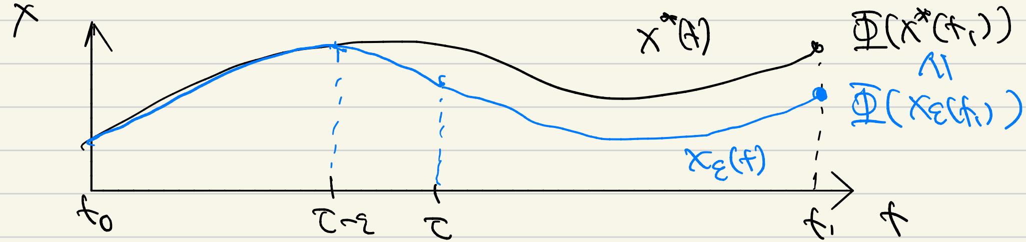
Fix $\tau \in (t_0, t_1)$, $s \in \Theta$, $\varepsilon > 0$.

Define $\underline{\theta_\varepsilon}$ with

$$\theta_\varepsilon(t) = \begin{cases} s & t \in [\tau - \varepsilon, \tau] \\ \theta^*(t) & \text{otherwise} \end{cases}$$



Define $\quad \dot{x}_\varepsilon(t) = f(t, x_\varepsilon(t), \theta(t)) \qquad x_\varepsilon(0) = x_0$



Step 3 : Variational Equation

Define $\quad v(t) = \lim_{\varepsilon \to 0^+} \dfrac{x_\varepsilon(t) - x^*(t)}{\varepsilon}. \qquad t \in [\tau, t_1]$

On $t \in [\tau, t_1]$, $\quad x^*, x_\varepsilon$ satisfies the same equation

$$\dot{x} = f(t, x, \theta^*)$$

$$\Rightarrow \quad \dot{v}(t) = \nabla_x f(t, x^*(t), \theta^*(t))\, v(t), \qquad v(\tau) = v_0.$$

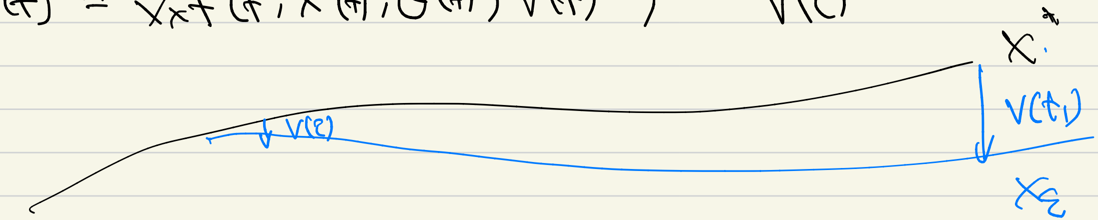
$$v(\tau) = v_0 = \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \left(\underbrace{x_0 + \int_0^\tau f(t, x_\varepsilon(t), \theta_\varepsilon(t))\, dt}_{x_\varepsilon(\tau)} - \underbrace{x_0 - \int_0^\tau f(t, x^*(t), \theta^*(t))\, dt}_{x^*(\tau)} \right) \right\}$$

y

$$= \lim_{\varepsilon \to 0^+} \left\{ \frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} f(t, x_\varepsilon(t), \delta)\, dt - \frac{1}{\varepsilon} \int_{\tau-\varepsilon}^{\tau} f(t, x^*(t), \theta^*(t))\, dt \right\}.$$

$$\overset{\tau\text{-a.e.}}{=} f(t, x^*(\tau), \delta) - f(t, x^*(\tau), \theta^*(\tau)) \qquad (\text{Lebesgue diff theorem}).$$

$$\dot{v}(t) = \nabla_x f(t, x^*(t), \theta^*(t))\, v(t), \qquad v(\tau)$$



Step 4: Optimality Condition.

$$\underbrace{v(t_1)}_{\text{depends on } \tau,\ \delta}{}^T \nabla \Phi(x^*(t_1)) \geq 0 .$$

$\Rightarrow$ Go to adjoint equation.

Γ $\dot{v}(t) = A(t)v(t)$ adjoint : $\dot{p}(t) = -A(t)^T p(t)$.

$$\frac{d}{dt}(v^T p)(t) = v^T \dot{p} + p^T \dot{v}$$
$$= v^T(-A^T)p + p^T A v = 0 .$$

$(v^T p)(t) =$ constant in t $\lrcorner$

Define co-state, or adjoint.

$$\begin{cases} p^*(t_1) = -\nabla\Phi(x^*(t_1)) \\ \dot{p}^*(t) = -\nabla_x f(t, x^*(t), \theta^*(t))^T p^*(t) \end{cases}$$

$$p^*(\tau)^T v(\tau) = p^*(t_1)^T v(t_1) \le 0 .$$

$$\Rightarrow \underbrace{p^*(\tau)^T f(\tau, x^*(\tau), s)}_{H} \le \underbrace{p^*(\tau)^T f(\tau, x^*(\tau), \theta^*(\tau))}_{H(\tau,\, x^*(\tau),\, p^*(\tau),\, \theta^*(\tau))}$$

Step 5: convert back to Bolza problem

$$\dot{p}^{*0}(t) = 0 \quad , \quad (p^*)^0(t_1) = -\nabla_{x^0} \tilde{\Phi}$$

$$= -\nabla_{x^0} \left(\Phi(x) + x^0 \right)$$

$$= -1.$$

$$\Rightarrow \quad p^{*0}(t) = -1 \quad \forall t.$$

$$\underbrace{p^*(\tau)^T f(\tau, x^*(\tau), s) - L(\tau, x^*(\tau), s)}_{H} \leq \cdots$$

$$\dot{x} = \theta = v$$

$$\int L(t, x, \dot{x})\, dt \qquad (\dot{x} \leftarrow v)$$

$$\dot{x} = v$$

$$\dot{x} = f(t, x, \theta)$$

$$\boxed{H(t, x, p) = \sup_{v} \left(p^T v - L(t, x, v) \right)}$$

$$\boxed{H(t, x, p, v) = \sup_{v} p^T v - L(t, x, v)}$$

$$H(t, x, p^*, v^*)$$

Bolza Problem with fixed end point, variable time.

$$\inf_{\underline{\theta}} J[\underline{\theta}] = \int_{t_0}^{t_1} L(t, x(t), \theta(t))\, dt \; \cancel{+ \; \Phi(x(t_1))}$$

$$\text{subj to} \quad \dot{x} = f(t, x, \theta), \quad t \in [t_0, t_1], \quad x(t_0) = x_0$$

$$x(t_1) = x_1$$

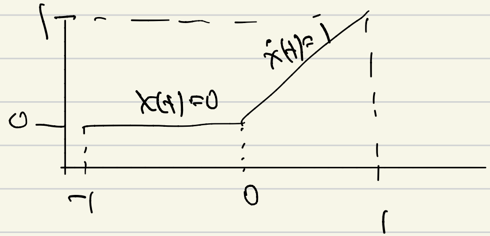
PMP.

$$\begin{cases} \dot{x}^* = \nabla_p H(t, x^*, p^*, \theta^*) & x^*(t_0) = x_0 \quad x^*(t_1) = x_1 \\ \dot{p}^* = -\nabla_x H(t, x^*, p^*, \theta^*) & \cancel{p^*(t_1) = -\nabla \Phi(x^*(t_1))} \\ H(t, x^*(t), p^*(t), \theta^*(t)) \geq H(t, x^*(t), p^*(t), \theta) & \forall \; \theta \in \Theta \end{cases}$$

Example

$$\min_{\underline{x}} \int_{-1}^{1} x(t)^2(\theta(t)-1)^2\,dt \quad , \quad \underset{x_0}{x(-1)=0}, \quad \underset{x_1}{x(1)=1}$$

$$\dot{x}(t) = \theta(t)$$



$$x^*(t) = \begin{cases} 0 & 1 \le t \le 0 \\ t & 0 < t \le 1 \end{cases}$$

$\underline{x}^*$ is global minimizer.

$$L(t,x,\theta) = x^2(\theta-1)^2 \qquad H(t,x,p,\theta) = p\theta - x^2(\theta-1)^2.$$

PMP: $\dot{x}^*(t) = \theta^*(t) \qquad x^*(-1)=0, \quad x^*(1)=1$

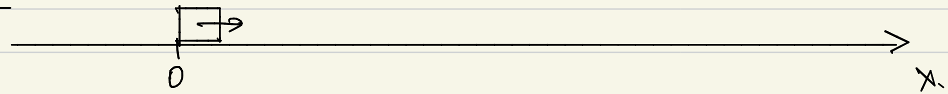
$$\dot{p}^*(t) = 2x^*(t)(1-\theta^*(t))^2$$

$$\theta^*(t) \in \arg\max_{\theta} \left\{ p^*(t)\,\theta - [x^*(t)]^2(1-\theta^2) \right\}.$$

$$\theta^*(t) = \begin{cases} 0 & 1 \le t \le 0 \\ 1 & 0 < t \le 1 \end{cases} \qquad p^*(t) = 0 \ \forall t.$$

Example

$x(t)$ position of car at time t.



$\ddot{x}(t) = \theta(t)$ $\qquad$ $\theta(t) \in [-1, 1]$

$$\inf_{\theta} J[\theta] = -x(T) + \int_0^T \frac{1}{2} \max(0, \theta(t))^2 \, dt$$

$\max(0, \theta(t))$

$$\dot{x} = v \qquad x(0) = 0. \qquad \theta(t) \in \Theta = [-1, 1]$$
$$\dot{v} = \theta. \qquad v(0) = 0.$$

Solution using PMP.

$$H(t, x, v, p_x, p_v, \theta) = \begin{pmatrix} p_x \\ p_v \end{pmatrix}^T \begin{pmatrix} v \\ \theta \end{pmatrix} - \frac{1}{2} \max(0, \theta)^2.$$

$$= p_x v + p_v \theta - \underline{\frac{1}{2} \max(0, \theta)^2}$$

PMP Equations

$$\begin{cases} \dot{x}^* = v^* & \qquad x^*(0) = 0 \\ \dot{v}^* = \theta^* & \qquad v^*(0) = 0 . \\ \dot{p}_x^* = 0 & \qquad p_x^*(T) = 1 \\ \dot{p}_v^* = -p_x^* & \qquad p_v^*(T) = 0 . \end{cases}$$

$$\Rightarrow \quad p_x^*(t) = 1 \quad \forall t \quad \Rightarrow \quad p_v^*(t) = T - t .$$

$$H(t, x^*(t), v^*(t), p_x^*(t), p_v^*(t), \theta) = v^*(t) + \theta(T-t) - \frac{1}{2}\max(0,\theta)^2 .$$

$$\theta^*(t) = \underset{\theta \in [-1,1]}{\operatorname{argmax}} \; H \qquad\qquad \theta = \underline{T-t'} \longrightarrow$$

$$= \min(T-t, 1)$$

$T=5$