Conformally Invariant Loop Measures

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Abstract

There have been incredible progress in the last twenty years in the rigorous analysis of planar statistical mechanics models whose limits are conformally invariant. This paper will not try to survey all the recent advances. Instead, it will discuss some recent results about particular conformally invariant measures on loops and paths.

1 Introduction

One of the main goals in statistical physics is to understand macrosopic behavior of a system given the interactions which are mainly microscopic but may exhibit long range correlations. Such models often depend on a parameter and at a critical value of the parameter the collective interaction switches from being microscopic to macroscopic. Critical phenomena is the study of such systems at or near this critical value.

There is a wide class of models (percolation, self-avoiding walk, Ising and Potts model, loop-erased random walk and spanning trees,...) whose behavior is very dependent on the spatial dimension. There exists a critical dimension above which the behavior is relatively simple (although it is not always trivial to prove this is true!), but below the critical dimension there is "non mean-field" behavior with nontrivial critical exponents for long-range correlations and fractal structures arising.

It was first predicted by Belavin, Polyakov, and Zamolodchikov [3, 2] that the continuum limit of critical fields in two dimensions would exhibit some kind of conformal invariance. This idea along with the related Coulomb gas techniques allowed for a number of nonrigorous predictions of critical exponents, see, e.g., [8, 7, 10, 48, 49, 51]. These exact exponents agreed with simulations so even though the theoretical arguments were far from being mathematically rigorous, it seemed clear that they were giving correct predictions and hence there should be mathematical structures and theorems to make precise and prove these predictions.

Major breakthroughs in the rigorous theory happened in around the turn of the twenty-first century. Probably the most important is Schramm's creation of what it now called the Schramm-Loewner evolution (SLE). This combined with ideas of Werner and myself on the Brownian intersection exponent opened up the understanding of the continuum limit for curves and interfaces of fields. On the discrete side, Kenyon used conformal invariance to prove the exact value of the dimension of the loop-erased walk and Smirnov proved "Cardy's formula" for the crossing probabilities of critical percolation on the triangular lattice.

These works were just a start to what may be called a major subfield studying critical behavior of two-dimensional systems. This has included two Fields medals [47, 21], two other plenary talks [53, 37], at least four previous invited talks [56, 25, 13] plus a number of other invited talks somewhat related, and it has been a part of the work of at least four invited speakers in this conference.

Given the explosive nature of the field, I will not try to give an overview. I have decided to give a personal perspective and to focus on several loop measures and related models, loop-erased random walk (related to uniform spanning trees) and the Gaussian free field. I start by introducing one of the main characters, discrete loop measures, and show how they are related to some well known objects, spanning trees and

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determinant of the Laplacian. It also generates one of the random fractals, the loop-erased random walk, and we then discuss what it means to take a scaling limit. This leads to a review of two of the main players in the field: the conformally invariant Brownian loop measure and the Schramm-Loewner evolution (SLE). I discuss a number of properties of SLE and focus on the most recent part to finish the characterization, the natural or fractal parametrization of the curve.

The Gaussian free field in two dimensions is the next topic. I start with the discrete field and show recent results that construct the field using the loop measure with some extra randomness. I then introduce the continuous free field which has become the centerpiece of much of the work in conformally invariant systems. Here I only give a quick introduction. In respect for my advisor, Ed Nelson, I have decided to phrase this section in terms of nonstandard analysis [46]. While I am not sure this will add to the mathematical development of the field, I have a hope that it will be a pedagogical tool in the future to explain the relationship between the discrete and the continuous. Here I use it to help define "Liouville quantum gravity" which is the exponential of the Gaussian field.

The next part of the paper concerns the second type of loop, SLE type loops. I again give a discrete introduction focusing on the loop-erased random walk and spanning tree model. I then discuss recent constructions of such loops — either as part of conformal loop ensembles (CLE) or directly from the definition of SLE. The first construction uses the Brownian soup directly and the second construction is modeled on the definition of the Brownian loop measure and makes use of the natural parametrization.

I finish my discussing some recent results that combine the ideas in this survey, the convergence of the loop-erased walk to SLE_2 in the natural parametrization. This requires a combinatorial estimate involving a signed loop measure and reduces the discrete problem to a calculation for the Brownian loop measure. Then, it is shown how this relates to the the natural parametrization of SLE.

2 Loop measures and spanning trees

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We start with a simple definition. Given a countable set $\mathcal{X}$ and a function $p : \mathcal{X} \times \mathcal{X} \to [0, \infty)$, we define a *(rooted) loop* $l = [l_0, l_1, \ldots, l_n]$ to be a finite sequence with $l_j \in \mathcal{X}$ and $l_0 = l_n$. An important case is when p is the transition matrix for a Markov or subMarkov chain. We let $|l| = n$ denote the number of vertices in the loop; if $|l| = 0$, the loop is *trivial* and $p(l) = 1$; if $|l| > 0$, $p(l) = p(l_0, l_1)\, p(l_1, l_2) \cdots p(l_{n-1}, l_n)$. Our loop measures will always be on nontrivial loops.

An *(unrooted)* loop ℓ is an equivalence class of rooted loops under the equivalence

$$[l_0, l_1, \ldots, l_m] \sim [l_1, \ldots, l_m, l_1] \sim [l_2, l_3, \ldots, l_m, l_1, l_2] \sim \cdots .$$

In other words, an unrooted loop is a rooted loop that has retained its orientation but has forgotten its starting point. We can write $p(\ell)$ since $p(l)$ is the same for all representatives. The number of representatives of an unrooted loop divides n but can be strictly less than n; for example, the unrooted loop generated by $[x, y, x, y, x]$ has only two representatives. The loop measure on rooted loops is given by

$$\tilde{m}(l) = \tilde{m}_p(l) = \frac{p(l)}{|l|},$$

and the loop measure on the unrooted loops is the induced measure

$$m(\ell) = m_p(\ell) = \sum_{l \in \ell} \tilde{m}(l) = \frac{s(\ell)\, p(\ell)}{|\ell|},$$

where $s(\ell)$ denotes the number of rooted representatives of ℓ. In this generality, there is no need to restrict to positive values of p; indeed it can be complex-valued or even matrix-valued.

As an example, we will assume that $\mathcal{X}$ is a finite, connected, (undirected) graph and that p gives the transition probabilities for simple random walk on the graph. In other words, $p(x, y) = 1/d_x$ if x is adjacent to y where d_x is the degree of x. Wilson [61] found the following algorithm for choosing a spanning tree from the uniform distribution over all spanning trees. Let us write $\mathcal{X} = \{x_0, x_1, \ldots, x_n\}$ where we have chosen an arbitrary ordering of the vertices. We choose a spanning tree as follows thinking of x_0 as the root vertex:

- Start a random walk at x_1 and stop it when it reaches x_0 and erase the loops chronologically from the path. Add these edges to the tree.
- Recursively, choose the vertex of smallest index that has not been added to the tree; start a random walk there and stop it when it reaches a vertex in the tree; erase loops and add those edges to the tree.

We continue until we have a spanning tree. A straightforward analysis of the algorithm (see [30, Chapter 9]) shows that the probability that a particular tree $\mathcal{T}$ is chosen is

$$\left[\prod_{j=1}^{n} p(y_j, \hat{y}_j)\right] F(A) = \left[\prod_{y \in \mathcal{X} \setminus \{x_0\}} d_y\right]^{-1} F(A), \quad F(A) := \prod_{j=1}^{n} G_{A_j}(y_j, y_j).$$

Here $\{y_1, \ldots, y_n\}$ is a permutation of $A := \{x_1, \ldots, x_n\}$ (determined by $\mathcal{T}$); $\hat{y}_j$ is the vertex adjacent to y_j in $\mathcal{T}$ on the path to x_0; $A_j = A \setminus \{y_1, \ldots, y_{j-1}\}$; and G_{A_j} denotes the usual random walk Green's function for the walk killed upon leaving A_j. The term in brackets is clearly independent of the permutation. While it is not obvious that our definition for $F(A)$ does not depend on the ordering of the vertices, it indeed does not. One can check this as a simple exercise in Markov chain theory but it is more illuminating to write it in one of two order independent ways:

- $F(A) = 1/\det \Delta$ where $\Delta = G^{-1} = (I - P)$ is the (negative of the random walk) Laplacian considered as a matrix indexed by A_1.
- If $m = m_p$,

$$F(A) = \exp\left\{\sum_{\ell \subset A} m(\ell)\right\}, \tag{1}$$

The surprising fact is that Wilson's algorithm gives equal probability to each spanning tree; moreover, since we know what this probability is we can conclude that the number of spanning trees is

$$\left[\prod_{y \in \mathcal{X} \setminus \{x_0\}} d_y\right] F(A)^{-1} = \left[\prod_{y \in \mathcal{X} \setminus \{x_0\}} d_y\right] \det \Delta.$$

This looks even nicer if we use the graph Laplacian Δ_g (the degree matrix minus the adjacency matrix) in which case the right-hand side becomes just $\det \Delta_g$. This is far from being a new result — it was proved by Kirchhoff in the nineteenth century.

The fact that the quantity in (1) is a determinant can be seen if we write it in terms of the *rooted* loop measure and use a well known identity,

$$\exp\left\{\sum_{l \subset A_1} \hat{m}(l)\right\} = \exp\left\{\sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}(P^n)\right\} = \det[I - P].$$

The great utility of the loop measure comes from its description in terms of *unrooted* loops; indeed, the proof of Wilson's algorithm uses the fact that one sample from a "soup" of unrooted loops in any order.

Although this can be done in generality, we will be focusing on a special case. Suppose that A is a finite, simply connected subset of the integer lattice $\mathbb{Z}^2$ containing the origin so that $\partial A = \{x \in \mathbb{Z}^2 : \operatorname{dist}(x, A) = 1\}$. The usual simple random walk measure gives $p(x, y) = 1/4$ if $|x - y| = 1$. Suppose we take a simple random walk starting at the origin, stop it when it reaches ∂A, and then erase loops to give a self-avoiding path η. This gives a probability measure on self-avoiding walks (SAW) starting at the origin ending and ∂A. By

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Wilson's algorithm, it is the probability that the unique path from the origin to ∂A in the uniform spanning tree of the graph $\mathcal{X} = A \cup \{\partial A\}$ is η (here, ∂A is considered as a single point — this is called the wired spanning tree). In this case, the number of spanning trees is $4^{\#(A)}\, F(A)^{-1}$.

Associated to loop measures are *loop soups*. This is a colorful term for a Poissonian realization from m. Let $\mathcal{L}_A$ denote the set of unrooted loops in the set A. At each time t, the soup $\mathcal{C}_t(A)$ is a multiset from $\mathcal{L}_A$ where loop ℓ appears N_t^ℓ times. It is defined by saying that $\{N_t^\ell : \ell \in \mathcal{L}_A\}$ are independent Poisson processes with parameter $m(\ell)$.

There are various ways to describe the probability distribution for loop-erased random walk from the origin to ∂A without making reference to loop erasure. One nice one is as a *Laplacian random walk*. Suppose the path starts as $\eta = [\eta_0 = 0, \eta_1, \ldots, \eta_k]$. Then the probabilities for the next step are given by weighting by the solution of the Dirichlet problem (for the discrete Laplacian) in $A \setminus \eta$ with boundary value 0 on η and 1 on ∂A. In other words, loop-erased random walk is Laplacian growth where the growth only occurs at the tip.

Suppose we observe the loop-erased walk η. Can we recover (with added randomness) the simple random walk that produced η? The answer is yes, and the way to do it is by taking a realization of the loop soup $\mathcal{C}_1$ at time $t = 1$. We then use η to "explore" the loop soup. We start at the origin and view all loops in $\mathcal{C}_1$ that intersect the origin. We turn these into rooted loops by choosing the origin as the root (if the origin is visited several times choose randomly) and then add all the loops to the path in the order they appeared in the soup. At this point we have not observed the soup in $A \setminus \{0\}$. We take our next step η_1 and observe the loops in $A \setminus \{0\}$ that intersect η_1, and continue. A short combinatorial argument [30, Chapter 9] shows that the distribution of the path at the end is that of a usual simple random walk. Note that the order in which we discover loops in the soup depends on the choice of η, and for a particular η we only observe the loops that intersect η.

The probability that a particular η is chosen for the loop-erased walk in A is $4^{-|\eta|}\, F_\eta(A)$, where $\log F_\eta(A)$ denotes the loop measure of loops in A that intersect η. What happens if we "perturb" the domain, say, consider $\tilde{A} \subset A$? How does this change the probability of seeing a certain η? This probability is zero if $\eta \cap (A \setminus \tilde{A})$ is nonempty, but otherwise it is $4^{-|\eta|}\, F_\eta(\tilde{A})$. In other words, the Radon-Nikodym derivative is given in terms of the loops in the larger domain that are lost when shrinking:

$$\frac{F_\eta(\tilde{A})}{F_\eta(A)} = \exp\left\{ - \sum_{\ell \subset A, \ell \cap (A \setminus A') \neq \emptyset, \ell \cap \eta \neq \emptyset} m(\ell) \right\}. \tag{2}$$

In continuous models for paths and loops, one of the key quantities to consider is the effect of perturbation of a domain on the measure of a particular path.

One observable of a loop soup is $n_x = n_x(t)$, the number of times that a vertex x is visited by time t. In the case $t = 1$ and the loop-erased walk, n_x is a geometric random variable,

$$\mathbf{P}\{n_x = k\} = q^k\,(1-q), \qquad q = \frac{1}{G_A(x,x)}.$$

More generally, the distribution of $n_x(t)$ is negative binomial,

$$\mathbf{P}\{n_x(t) = k\} = \binom{k+t-1}{k}\, q^k\,(1-q)^t.$$

One can convert to continuous times by adding independent exponential waiting times. For $t = 1$, given n_x, we define $L_x(1)$ to be the sum of $n_x + 1$ independent exponential random variables with parameter 1. Then a standard computation shows that $L_x(t)$ has an exponential distribution with parameter q. For other t, we can choose $L_x(t)$ to be the sum of $n_x(t)$ independent exponential random variables and a Gamma random variable with parameters t and 1. In particular, if $t = 1/2$, then $L_x(t)$ is the sum of $n_x(1/2)$ exponentials plus a random variable with the same distribution as $Z^2/2$ where Z is a standard normal.

As we discuss below, $\{L_x(1)/2 : x \in \mathcal{X}\}$ has the same distribution as $\{|Z_x|^2 : x \in \mathcal{X}\}$ where $\{Z_x = X_x + iY_x : x \in \mathcal{X}\}$ is a complex Gaussian field with independent real and imaginary parts each having covariance matrix G. Equivalently, $\{L_x(1/2)\}$ has the distribution of $\{X_x^2\}$.

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3 Scaling limits

In two dimensions, conformally invariant objects are obtained as scaling limits of discrete models. Both the loop soup and the loop-erased walks have limits that we describe here. To each finite, connected subset A of $\mathbb{Z}^2 = \mathbb{Z} + i\mathbb{Z}$ we associate a domain in $\mathbb{C}$,

$$D_A = \operatorname{int}\left[\bigcup_{z \in A} (z + \mathcal{S})\right], \quad \mathcal{S} = \{x + iy \in \mathbb{C} : |x|, |y| \leq 1/2\}.$$

Conversely, if D is a simply connected domain in $\mathbb{C}$ containing the origin and $1/n$ is a lattice spacing, we let A_n be the connected component containing the origin of all w with $w + \mathcal{S} \subset nD$. Then we set $D_n = n^{-1} D_{A_n}$ as a lattice approximation of the domain D. If $z \in n^{-1}\mathbb{Z}^2$, we write $\mathcal{S}_z = z + n^{-1}\mathcal{S}$.

As an example, let D be the square $\{x + iy : |x|, |y| < 1\}$ so that $A_n = \{x + iy \in \mathbb{Z}^2 : |x|, |y| < n\}$. Let $z_n = -n, w_n = n$. Then in the limit we get the square $D = \{x + iy \in \mathbb{C} : |x|, |y| < 1\}$ with the boundary points $z = -1, w = +1$. On A_n we have the LERW from z_n to w_n; to be more precise, we consider all simple random walk paths starting at z_n, ending at w_n, otherwise staying in A_n, and erase the loops chronologically. We also have the random walk loop measure. Here we consider the scaling limits of both.

We start with the loop measure which does not depend on the boundary points z, w. To each rooted loop $l = [l_0, \ldots, l_{2k}]$ in A_n we associate the scaled loop $l^{(n)}(t), 0 \leq t \leq k/n^2$,

$$l^{(n)}(j/n^2) = n^{-1} l_{2j}, \quad 0 \leq j \leq k, \tag{3}$$

extended to other t by linear interpolation. We give $l^{(n)}$ the same measure as l; in other words, the loop measure on D_n is the same as the loop measure on A_n except that the paths are scaled, As $n \to \infty$, the total mass of the measure goes to ∞. The Brownian scaling in (3) uses the relationship $dt = (dx)^2$ where 2 is the fractal dimension of Brownian paths. It is well known that the probability that a two-dimensional simple random walk is at its starting point after $2k$ steps is asymptotic to $(\pi k)^{-1}$, and hence the measure of loops $l^{(n)}$ rooted at a particular point $\zeta \in n^{-1}\mathbb{Z}^2$ of time duration $t = k/n^2$ is asymptotic to

$$\frac{1}{\pi k}\,\frac{1}{2k} = \frac{1}{2\pi t^2}\, dt \operatorname{Area}(\mathcal{S}_\zeta) \quad dt = \frac{1}{n^2}. \tag{4}$$

The scaling limit of this can now be determined [34] and is described in the next section.

The definition of the LERW from z to w starts with the usual random walk measure from z to w; to be more precise, every nearest neighbor walk $\omega = [\omega_0, \ldots, \omega_k]$ with $\omega_0 = -n, \omega_k = n$ and $\{\omega_1, \ldots, \omega_{k-1}\} \in A_n$ gets measure 4^{-k}. The total mass of this measure is the probability that a simple random walk starting at $-n$ immediately enters A_n and then exits at n. It can be shown (using, for example, the "gambler's ruin" estimate for random walk or in this case by explicit computation as a finite Fourier series) that the total mass is asymptotic to $c\,n^{-2}$ for a (computable) constant c. If we erase loops from the paths we get a measure on self-avoiding paths with the same total mass. As before, we can consider this as the measure on self-avoiding paths $\eta = [\eta_0 = -n, \ldots, \eta_k = n]$ with $\eta_1, \ldots, \eta_{k-1} \in A_n$, that gives measure $4^{-k} F_\eta(A_n)$, to each such η. The total mass can be written as the value of the *partition function*

$$Z_{A_n}(\beta; -n, n) := \sum_{\eta : -n \longrightarrow n, \eta \subset A_n} e^{-\beta|\eta|} F_\eta(A_n),$$

evaluated at the *critical* $\beta_c = \log 4$. The mass can also be described in terms of spanning trees. A wired spanning tree of A_n is a spanning tree of the graph $A_n \cup \{\partial A_n\}$ where all the points in the boundary have been identified to a single point. Using Wilson's algorithm rooted at the boundary with $x_1 = -n + 1$, we can see that the total mass is (1/4 times) the probability that the uniform spanning tree contains a path starting at $-n + 1$ and reaching the boundary along the edge adjacent to n.

In order to scale the paths, we need to know the fractal dimension d which can be defined roughly by saying the that typical LERW crossing A_n has on the order of n^d steps. Using the Brownian scaling as

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the model, we associate to each LERW $\eta = [\eta_0, \ldots, \eta_k]$ connecting $-n$ to n in A_n, the scaled path of time duration kn^{-d},

$$\eta^{(n)}(j/n^d) = n^{-1}\,\eta_j, \quad 0 \leq j \leq k. \tag{5}$$

When taking the limit, we first multiply the measure by n^2 so that the limit has finite total mass, and then we take a limit of the paths above. The limit is be a version of the Schramm-Loewner evolution (SLE).

4 Brownian loop measure and soup in $\mathbb{C}$

Let D be a bounded, simply connected domain in $\mathbb{C}$. For every n, let $A_n \subset \mathbb{Z}^2$ and $D_n = n^{-1} A_n$ be defined as in the previous section. The *Brownian loop measure* is the scaling limit as $n \to \infty$ of the random walk loop measure on A_n. It can be constructed directly [35], and, indeed, it was defined before the discrete loop measures.

A *rooted loop* is a curve $\gamma : [0, t_\gamma] \to \mathbb{C}$ with $\gamma(0) = \gamma(t_\gamma)$ where $t_\gamma \in (0, \infty)$. An *(unrooted) loop* is an equivalence class of rooted loops under the equivalence $\gamma \sim \gamma^r$ where $\gamma^r(t) = \gamma(t+r)$ and $t+r$ is interpreted modulo t_γ. We will define the loop measure first for rooted loops and then use this to define the measure for unrooted loops. It is useful to view a rooted loop γ as a triple $(z, t_\gamma, \hat{\gamma})$ where z is the root, t_γ is the time duration, and $\hat{\gamma}$ is a loop of time duration 1 rooted at the origin. The bijection is given using Brownian scaling

$$\gamma(t) = z + t_\gamma^{1/2}\,\hat{\gamma}(t/t_\gamma), \quad 0 \leq t \leq t_\gamma.$$

The rooted loop measure on $\mathbb{C}$ can be given by

$$(\text{Area}) \times \frac{dt}{2\pi t^2} \times (\text{Brownian bridge}),$$

Here Brownian bridge refers to the probability measure associated to (appropriately defined) Brownian motion starting at 0 conditioned to return to 0 at time one. The middle term can be written as $t^{-1}\,(2\pi t)^{-1}$. The factor $(2\pi t)^{-1}$ is the density of Brownian motion at time t evaluated at the origin and the t^{-1} is the analogue of the $|l|^{-1}$ term from the discrete loop measure. This is the natural continuum analogue of (4). To give the measure on a domain D one restricts this measure to loops in D. This is an infinite measure even for bounded D because the measure of small loops blows up. This measure induces a measure on unrooted loops which we call the *Brownian loop measure*. Poissonian realizations of the Brownian loop measure are called Brownian *loop soups*.

The Brownian loop measure satisfies two important properties. The first is immediate but still very important.

- **Restriction property**. If $D' \subset D$ then the loop measure on D' is the same as the loop measure on D restricted to loops that lie in D'.

The second is particular to two dimensions and is a starting point for analysis of conformally invariant processes.

- **Conformal invariance** [35]. If $f : D \to f(D)$ is a conformal transformation, then the image of the loop measure on D is the loop measure on $f(D)$.

Let us explain the last statement. The Brownian (heat equation) scaling can be written intuitively as $dx = \sqrt{dt}$. This is the scaling for a path with fractal dimension 2. (We will consider paths of fractal dimension d for which $dx = (dt)^{1/d}$.) The parametrization for the Brownian motion is a "natural" two-dimensional parametrization $|dB_t| \approx t^{1/2}$. As shown first by Lévy [40], complex Brownian motion is conformally invariant provided that one changes the parametrization to respect the fractal dimension. In our case, if γ is a loop in D of time duration t_γ, we define a loop $f \circ \gamma$ in $f(D)$ by

$$t_{f\circ\gamma} = \int_0^{t_\gamma} |f'(\gamma(s))|^2\,ds,$$

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and $f \circ \gamma(r) = f[\gamma(\sigma(r))]$ where

$$\int_0^{\sigma(r)} |f'(\gamma(s))|^2 \, ds = r.$$

If μ_D denotes the Brownian loop measure on *unrooted loops* and $f \circ \mu_D$ is defined by

$$f \circ \mu_D(V) = \mu_D\{\gamma : f \circ \gamma \in V\},$$

then $f \circ \mu_D = \mu_{f(D)}$. This result requires no topological assumptions on the domain D.

The definition of the loop measure is not very conducive to calculation. When computing measures of sets it is often useful to use a decomposition of into *Brownian (boundary) bubbles*. This focuses on a particular rooted representative of an unrooted loop. For example, if $\mu = \mu_{\mathbb{C}}$ is the measure on the entire plane, then we can write

$$\mu_{\mathbb{C}} = \int_{\mathbb{C}} \mu^{\text{bub}}_{\mathbb{H}+iy}(x + iy) \, dx \, dy.$$

This is a decomposition focusing on the (unique) point on the loop of smallest imaginary part. Here $\mu^{\text{bub}}_{\mathbb{H}}(0)$ is a σ-finite measure on loops rooted at the origin and otherwise staying in the upper half plane $\mathbb{H}$. It can be defined in a number of equivalent ways by taking limits. More generally, if $f : D \to f(D)$ is a conformal transformation, $z \in \partial D$, and z and $f(z)$ are analytic boundary points, we have the conformal covariance rule

$$f \circ \mu^{\text{bub}}_D(z) = |f'(z)|^2 \, \mu^{\text{bub}}_{f(D)}(z).$$

Another useful way to write μ_C is by focusing on the point of greatest magnitude

$$\mu_{\mathbb{C}} = \int_{\mathbb{C}} \mu^{\text{bub}}_{|z|\mathbb{D}}(z) \, dA(z), \tag{6}$$

where $\mathbb{D}$ is the unit disk.

5 Measures on self-avoiding curves

The loop-erased random walk is one of many lattice models for which scaling limits are expected to exist. Many of them are parts of more complicated fields, for example, loop-erased random walks arise as macrosopic paths in scaling limits of uniform spanning trees. Suppose D is a bounded, simply connected subdomain of $\mathbb{C}$ containing the origin which for ease we will assume has an analytic boundary. Let z, w be distinct points on the boundary. We will consider measures on (continuous) curves $\gamma : (0, t_\gamma) \to D$ with $\gamma(0-) = z, \gamma(t_\gamma+) = w$ (we sometimes allow $t_\gamma = \infty$). Much of the work of the last eighteen years has built on work of Oded Schramm [52] to understand the possible limits under the assumption that the limit is conformallly invariant or covariant. We will consider (positive) measures $\mu_D(z, w)$ on such curves; if $\|\mu_D(z, w)\| < \infty$, we write $\mu^{\#}_D(z, w)$ for the corresponding probability measure obtained by normalization. The goal is to find families of measures indexed by D, z, w which have the following properties.

- **Fractal dimension**. The measure is supported on simple curves of Hausdorff dimension $d \in [1, 2]$ with the appropriate fractal parametrization. If $f : D \to f(D)$ is a conformal transformation, then $f \circ \gamma$ is defined by $f \circ \gamma(t) = f[\gamma(\sigma_t)]$ where

  $$\int_0^{\sigma_t} |f'(s)|^d \, ds = t.$$

  Note that Brownian curves satisfy this with $d = 2$ but they are not simple. Although we start with an assumption of simplicity of the curves, it turns out that many important examples gives curves that are not simple. However, they will be "non-crossing". For the moment we restrict to the simple case.

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- **Conformal covariance**. If $f : D \to f(D)$ is a conformal transformation, then

$$f \circ \mu_D(z,w) = |f'(z)|^b \, |f'(w)|^b \, \mu_{f(D)}(f(z), f(w)),$$

where b is a scaling exponent. In particular the probability measures are conformally *invariant*:

$$f \circ \mu_D^{\#}(z,w) = \mu_{f(D)}^{\#}(f(z), f(w)),$$

and $\mu_D^{\#}(z,w)$ can be defined even for nonsmooth boundaries.

- **Reversibility**. The measure $\mu_D(w,z)$ is obtained from $\mu_D(z,w)$ by reversing the paths.

- **Boundary perturbation or generalized restriction**. Suppose $D' \subset D$ and the domains agree in neighborhoods of z, w. Then $\mu_{D'}(z,w)$ is mutually absolutely continuous with the measure given by $\mu_D(z,w)$ *restricted* to curves with $\gamma(0, t_\gamma) \subset D'$. If $\Phi_{D,D'}$ denotes the Radon-Nikodym derivative, then it is a conformal invariant,

$$\Phi_{D,D'}(\gamma) = \Phi_{f(D),f(D')}(f \circ \gamma).$$

- **Domain Markov property**. Suppose an initial segment $\tilde{\gamma}$ of γ ending at $z' \in D$ is observed. Then in the probability measure $\mu_D^{\#}(z,w)$, the conditional distribution of the remainder of the curve given $\tilde{\gamma}$ is given by $\mu_{D\setminus\tilde{\gamma}}(z', w)$. By reversibility, we should be able to also grow ends of the curve from w.

The big breakthrough by Schramm [52] described in our notation is as follows. Let us restrict to simply connected domains D, consider only the probability measures $\mu_D^{\#}(z,w)$ (which do not require boundary smoothness), and assume conformal invariance and the domain Markov property. Finally, consider the curves γ and $f \circ \gamma$ only *modulo reparametrization*. Then there is only a one parameter family of curves that are candidates for this. This is now called the *(chordal) Schramm-Loewner evolution (SLE)* with parameter $\kappa > 0$. It describes the curve $\gamma[0, t_\gamma]$ in terms of the collection of conformal maps $g_t : D \setminus \gamma[0,t] \to D$ with $g_t(\gamma(t)) = z, g_t(w) = w$. The evolution of g_t is described with a Loewner equation driven by a Brownian input. The parameter κ gives the variance of the Brownian motion. It takes some work to understand the curve $\gamma[0,t]$ from the maps g_t but we now know a lot.

- The measure is supported on simple curves for $\kappa \leq 4$; it is supported on plane-filling curves for $\kappa \geq 8$; and for $4 < \kappa < 8$, it is supported on self-intersecting but "non-crossing" curves that are not plane filling. [50]

- For $\kappa < 8$, the measure is supported on curves of Hausdorff dimension

$$d = 1 + \frac{\kappa}{8}.$$

In particular, for each $1 < d < 2$, there is a unique family of curves. [50, 1]

- The measure is reversible for $\kappa < 8$ [45, 64].

The relationship with the Brownian loop measure comes in the boundary perturbation rule. We define the conformal invariant: if D is a domain and K, K' are disjoint, relative closed, subsets, then $\Lambda_D(K,K') = \exp\{m(\mathcal{L})\}$, where $\mathcal{L} = \mathcal{L}_D(K,K')$ is the Brownian measure of loops in D that intersect both K and K'. Using mainly the work in [26], for $d \leq 4$ we can define the measure $\mu_D(z,w)$ such that the following is true.

- The total mass of $\mu_D(z,w)$ is $H_D(z,w)^b$ where $b = \frac{6-\kappa}{2\kappa}$, and $H_D(z,w)$ denotes the boundary Poisson kernel (the normal derivative in each component of the Green's function) normalized so that $H_{\mathbb{H}}(0,1) = 1$.

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- If $D' \subset D$ as above and $\gamma(0,t_\gamma) \subset D'$, the Radon-Nikodym derivative is given by

$$\frac{d\mu_{D'}(z,w)}{d\mu_D(z,w)}(\gamma) = \Lambda_D(\gamma, D \setminus D')^{\mathbf{c}/2} = \exp\left\{\frac{\mathbf{c}}{2}\, m[\mathcal{L}_D(\gamma, D \setminus D')]\right\}, \tag{7}$$

  where $\mathbf{c}$ is the *central charge* given by

$$\mathbf{c} = \frac{(3\kappa - 8)(6-\kappa)}{2\kappa}.$$

  This is the same central charge that is a fundamental parameter in conformal field theories. In statistical physics, it also can be described in terms of infinitesimal changes of the "stress energy tensor". Here we see it as measuring the effect on the path measure of infinitesimal changes to the ambient domain.

The scaling limit of LERW is SLE_2. (This was predicted in [52] and proved, at least for the related radial case, in [32] for curves modulo reparametrization.) Here we see $\mathbf{c} = -2$; indeed (7) is the scaling limit of the relation (2). This answers all of the questions above except for giving the path the correct parametrization; we discuss this in Section 11.

The theory of chordal SLE in simply connected domains can be derived from the assumptions of conformal invariance and domain Markov property of the probability measures $\mu_D^{\#}(z,w)$. In fact, the role of the partition function and the Brownian loop measure was found by studying the unique one-parameter family of measures in simple connected domains. There is a another way of looking at this that is important. Let us consider the case of the upper half plane and boundary points 0 and infinity. The fundamental observation of Schramm was the following. Suppose that we consider the case of the upper half plane $\mathbb{H}$ and boundary points 0 and infinity. One version of "Loewner chains" (which were developed by Loewner to understand the Bieberbach conjecture) states that if γ is a simple curve from 0 to ∞; $H_t = \mathbb{H} \setminus \gamma[0,t]$, and $g_t : H_t \to \mathbb{H}$ is the unique conformal transformation satisfying

$$g_t(z) = z + o(1), \quad z \to \infty,$$

then *with an appropriate parametrization of* γ, there is a continuous function real-valued function U_t such that

$$\partial g_t(z) = \frac{2}{g_t(z) - U_t}, \quad g_0(z) = z.$$

Schramm noted that conformal invariance and the domain Markov property implied that U_t is a driftless Brownian motion, and hence $U_t = \sqrt{\kappa}\, W_t$, where W_t is a standard Brownian motion and κ is the parameter. Setting $a = 2/\kappa$ and doing a linear time change, we get

$$\partial g_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z,$$

where $U_t = -B_t$ is a standard Brownian motion. If $Z_t(z) = g_t(z) - U_t$, we can write this as a Bessel equation,

$$dZ_t(z) = \frac{a\, dt}{Z_t(z)} + dB_t.$$

For $\kappa \leq 4$, this gives a measure on simple curves and the following is true. Suppose $D = \mathbb{H} \setminus K$ is a simply connected subdomain where K is compact, not containing the origin. There are three equivalent ways to find $\mu_D^{\#}(0,\infty)$.

- Use a conformal transformation to map $\mathbb{H}$ onto D fixing 0 and ∞ and use conformal invariance.

- Give the Radon-Nikodym derivative of the two probability measures. Assuming $\gamma \cap K = \emptyset$, the value is

$$\Phi_D'(0)^{-b} \exp\left\{\frac{\mathbf{c}}{2}\, m[\mathcal{L}_D(\gamma, D \setminus D')]\right\}.$$

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where $\Phi_D : D \to \mathbb{H}$ with $\Phi_D(0) = 0, \Phi_D(\infty) = \infty, \Phi_D'(\infty) = 1$. Equivalently, we can define the measure $\mu_D(0, \infty)$ with total mass (partition function)

$$\Phi_D'(0)^b := \mathbf{E}\left[\exp\left\{\frac{\mathbf{c}}{2}\, m[\mathcal{L}_D(\gamma, D \setminus D')]\right\}\right],$$

and this satisfies the generalized restriction property.

- Give the Radon-Nikodym derivative on the probability space on which the Brownian motion is defined. One standard way to construct "adapted" absolutely continuous measures to a Brownian motion is to give a drift,

$$dY_t = R_t\, dt + dB_t,$$

where R_t is measurable with respect to the process at time t. The process in D is SLE in all of $\mathbb{H}$ "weighted by" or "tilted" by the partition function. The precise meaning of this is given by the Girsanov theorem. Let M_t be the partition function of the remaining domain seen at time t; more precisely, M_t is a (local) martingale of the form

$$M_t = C_t\, \Phi'_{g_t(D)-U_t}(0)^b,$$

where C_t is a differentiable process that can be considered as a continuous normalization to be a probability measure. Then if we change the probability measure to weight by M_t, then

$$dB_t = R_t\, dt + d\tilde{B}_t, \qquad R_t = b\,[\log \Phi'_{g_t(D)-U_t}(0)]'$$

where $\tilde{B}_t$ is a standard Brownian motion in the new measure. An equivalent way to specify the process in D is to give the drift term R_t which is the logarithmic derivative of the partition function.

One of the main reasons that the conformal Markov property (domain Markov property and conformal invariance) determine SLE up to a single parameter, is the fact that a domain obtained from slitting a simply connected domain from the boundary is still simply connected and hence conformally equivalent to the original domain. This is not true for nonsimply connected domains and more general Riemann surfaces. Extending SLE to more general domains requires making more assumptions than just the conformal Markov property. One possibility is to use the Brownian loop measure and use the generalized restriction measure to define the measure $\mu_D(z, w)$ for other domains. This will satisfy the conformal Markov property but is not the unique process. Another idea is to find the partition functions $\Phi_D(z, w)$; if one does (perhaps as limits of some discrete model) and can prove sufficient smoothness, then one can define the process in terms of the logarithmic derivative. Finding the correct partition function for a scaling limit of a model is a big step to understanding the behavior.

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6 Natural parametrization

The use of the Loewner differential equation to study SLE requires parametrization by some form of capacity, that is, by the size of the set seen by a Brownian motion starting away from the set. For example, if we consider SLE from 0 to infinity in the upper half plane $\mathbb{H}$ with corresponding maps g_t, then the parametrization is such that for $z \in \mathbb{H}$, $t \mapsto g_t(z)$ is continuously differentiable.

The scaling limit of discrete curves such as in (5) should also give a parametrization of the curves. It turns out that not only is this not the same measure, it and the capacity parametrization are *singluar* with respect to each other. However, one can use the properties of SLE to find this new parametrization which is sometimes called *natural parametrization*. It is the fractal d-dimensional analogue of parametrization by arc length. Brownian paths have the natural 2-dimensional parametrization.

Given a path γ, the natural parametrization would be defined so that the "d-dimensional length" of $\gamma[0,t]$ is t. One well-known d-dimensional measure is Hausdorff measure defined (at least up to a constant) by

$$\mathcal{H}^d(V) = \lim_{\epsilon \downarrow 0} \inf \sum_{j=1}^{\infty} [\mathrm{diam}\, U_j]^d,$$

where the infimum is over all covers of V by sets of diameter at most ϵ. The Hausdorff dimension of V is the unique d at which $\mathcal{H}^d(V)$ jumps from ∞ to 0; the value at d can be anything. For random sets of Hausdorff dimension d, typically we have $\mathcal{H}^d(V) = 0$. Roughly speaking, this is because we can take covers by sets of any size less than or equal to ϵ, and given a realization of the random set, the optimal cover takes advantage of this freedom. There are refinements of Hausdorff measure using gauge functions, and the optimal gauge function is well understood for some processes such as Brownian motion. However, for processes with very strong dependence on the immediate past such as SLE, determining a correct gauge correction is difficult and open.

To parameterize SLE paths it is more useful to take a naive approach and try to cover by balls of radius ϵ; this is much closer to the approximation by a lattice since one has a fixed lattice size. A similar idea is the *d-dimensional Minkowski content* which for subsets V of $\mathbb{C} = \mathbb{R}^2$ is given by

$$\mathrm{Cont}_d(V) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathrm{Area}\{z : \mathrm{dist}(z, V) \leq \epsilon\}, \tag{8}$$

provided that this limit exists. Rezaei and I [31] were able to show that this limit exists and is nontrivial for $SLE_\kappa, \kappa < 8$ (for $\kappa \geq 8$ the curve is plane filling and the natural parametrization should be parametrization by area). In particular, the curve γ can be reparametrized such that for each s, $\mathrm{Cont}_d[\gamma[0,s]) = s$.

Proving the existence of this limit starts with hoping that it exists and seeing what this would imply. Consider SLE from z to w in a domain D and let $\zeta \in D$. The (chordal SLE) Green's function $G_D(\zeta; z, w)$ is the normalized probability that the SLE path goes through ζ, more precisely,

$$G_D(\zeta; z, w) = \lim_{\epsilon \downarrow 0} \epsilon^{d-2}\, \mathbf{P}\{\mathrm{dist}(\zeta, \gamma) \leq \epsilon\}.$$

Establishment of the limit on the right-hand side is essentially the same as showing that for fixed $0 < \rho < 1$ as $\epsilon \to 0$,

$$\mathbf{P}\{\mathrm{dist}(\zeta, \gamma) \leq \rho\epsilon \mid \mathrm{dist}(\zeta, \gamma) \leq \epsilon\} \sim \rho^{2-d}.$$

This requires understanding the distribution of the tip of γ when it first gets within ϵ of ζ.

For simply connected D, it was noted in [50] that if such a function existed, then $M_t := G_{D \setminus \gamma_t}(\zeta; \gamma(t), w)$ would have to be a local martingale and an Itô's formula calculation gave $d = 1 + \frac{\kappa}{8}$ and

$$G_D(\zeta; z, w) = r_D(\zeta)^{d-2}\, S_D(\zeta; z, w)^{\frac{8}{\kappa} - 1}, \tag{9}$$

where $r_D(\zeta)$ denotes conformal radius and $S_D(\zeta; z, w)$ denotes the sine of the (conformally invariant) argument of ζ with respect to z, w. Here $\gamma_t = \gamma[0,t]$. Having made the observation, we can use the local martingale given by the left-hand side of (9) and the Girsanov theorem to understand the local behavior of the path as it gets near ζ. To establish the Minkowski content, one needs to improve this to a "two-point" estimate,

$$\mathbf{P}\{\mathrm{dist}(\zeta, \gamma) \leq \rho\epsilon, \mathrm{dist}(\zeta', \gamma) \leq \rho\epsilon \mid \mathrm{dist}(\zeta, \gamma) \leq \epsilon, \mathrm{dist}(\zeta', \gamma) \leq \epsilon\} \sim \rho^{2(2-d)}.$$

The natural parametrization satisfies a kind of Markovian property. Suppose D is a bounded domain, z, w are distinct boundary points, and $\gamma(t)$ is an SLE_κ path from z to w in D. Let $\Theta_t = \mathrm{Cont}_D(\gamma_t)$. Then

$$\mathbf{E}[\Theta_\infty] = \int_D G_D(\zeta; z, w)\, dA(\zeta).$$

$$\mathbf{E}\left[\Theta_\infty \mid \gamma_t\right] = \Theta_t + \Psi_t, \quad \Psi_t := \int_{D_t} G_{D_t}(\zeta; \gamma(t), w)\, dA(\zeta) \tag{10}$$

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where $D_t = D \setminus \gamma_t$. Since $\mathbf{E}[\Theta_\infty \mid \gamma_t]$ is a martingale, we can characterize Θ_t as the unique increasing process such that $\Psi_t + \Theta_t$ is a martingale (Doob-Meyer decomposition). The first construction [33, 36] of the natural parametrization used this characterization and it is important in the proof of the discrete parametrization of LERW to the natural parametrization of SLE_2.

Another way of viewing a "d-dimensional" parametrization is in terms of the Hölder exponent. Under the natural parametrization, the SLE_κ curves are Hölder continuous for all $\alpha < 1/d$ [62].

7 Gaussian field

Maybe the most fundamental random field is the *Gaussian (free) field*, that is, variables $\{Z_x : x \in \mathcal{X}\}$ indexed by a set which can be finite, countable, or uncountable, such that each finite dimensional distribution is multivariate Gaussian. The distribution is determined by the means and the covariances and we say it is centered if the means are zero. A relationship between random paths and Gaussian fields has been known for a while, (see, e.g., [6, 15, 57]) but what we describe here relating to loop measures is more recent due to Le Jan [38, 39] and Lupu [41].

We started with a discrete-time, discrete-space loop measure and then described the Brownian loop measure which is continuous-time, continuous space. We will also consider continuous-time, discrete space. There are two ways to get a loop measure with continuous times on a discrete space. Le Jan's approach is to use a definition analogous to Brownian loop measure by having paths from a continuous time Markov chain. The other is to start with discrete time loops and then add waiting times. Both approaches have advantages; we will use the latter approach here. Suppose we have a finite set $\mathcal{X}$ and a real-valued symmetric function q on edges; for ease, we will assume $q(x,x) = 0$ although the definitions here can be adapted to allow for self-edges. Such a weight gives a measure on paths by multiplying the weight of the edges and hence also gives a weight on loops. We will assume that this weight is actually a measure

$$\sum_\omega |q(\omega)| < \infty.$$

where the sum is over all finite length paths in A. In particular the Green's function can be written as

$$G(x,y) = [I - Q]^{-1}_{x,y} = \sum_{\omega : x \to y} q(\omega).$$

A particular case is when q are the transition probabilities for a subMarkov chain. There corresponds a weight on rooted loops $m(l) = m_q(l) = q(l)/|l|$ and the corresponding measure on unrooted loops. The centered Gaussian field $\{Z_x : x \in \mathcal{X}\}$ with covariance matrix G is the random vector whose Radon-Nikodym derivative with respect to independent, standard Gaussians is

$$\sqrt{\det(I-Q)}\, \exp\left\{\sum_e q_e\, Z_e\right\}$$

where the sum is over all undirected edges $e = \{x,y\}$ and $Z_e = Z_x Z_y$. If we consider the random field $\bar{T} = \{T_x = Z_x^2/2; x \in \mathcal{X}\}$, then the density of $\bar{T}$ can be written as

$$g(\bar{t})\, \sqrt{\det(I-Q)}\, \mathbf{E}\left[\exp\left\{\sum_e 2\, J_e\, q_e \sqrt{t_e}\right\}\right]$$

where $g(\bar{t})$ is the density for independent $\chi_1^2/2$ random variables; the sum is over all edges $e = \{x,y\}$, $t_e = t_x t_y$, $J_e = J_x J_y$; and $\{J_x\}$ are independent with $\mathbf{P}\{J_x = \pm 1\} = 1/2$.

To get a realization of $\bar{T}$ we can proceed as follows.

- Start with a realization of $\bar{T}$ for independent standard normals, that is, $\{t'_x\}$ independent with $\chi_1^2/2$ distributions.

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- Take a realization of the discrete loop soup giving local times $\{n_x\}$.
- Replace each n_x with the sum of n_x independent exponentials with rate 1 and add this to $\{t'_x\}$ to get $\{t_x\}$.

There are several ways to verify it; in [29], motivated by [42], it was done in a way to also get the joint distribution with the distribution on currents, that is, functions $\bar{k}$ on edges with the property that each vertex has an an even number of edges coming out of it. We start with a realization of the loop soup from m with intensity 1/2. If $q \geq 0$, this induces a probability distribution on currents $\bar{k}$. Given $\bar{k}$, we also get the local times n_x on vertices (n_x is 1/2 times the number of edges in $\bar{k}$ that intersect x). A little work shows that the probability that the current $\{k_e : e \in \mathcal{E}\}$ with corresponding local times $\{n_x : x \in \mathcal{X}\}$ is chosen is

$$\sqrt{\det(I-Q)}\left[\prod_{x\in\mathcal{X}} \frac{\Gamma(n_x+\frac{1}{2})}{\sqrt{\pi}}\right]\left[\prod_{e\in\mathcal{E}} \frac{\theta_e^{k_e}}{k_e!}\right], \quad \theta_e = 2q_e.$$

This formula works for nonpositive integrable weights if we interpret the Poisson process as the measure on loops. With this the validity of the claim above is straightforward to verify.

The loop soup and some extra randomness give the square of the field Z_x^2, but that is insufficient for determining the signs. For positive weights, the signs of the field can be chosen so that the clusters formed by the loop soup are all of the same sign. However, some more randomness is needed. Suppose $V \subset \mathcal{X}$ and let G_V, G_{V^c} be the corresponding Green's functions for the random walks restricted to those sets. Then the density of the random field on $\mathcal{X}$ with respect to independent fields on V and $\mathcal{X} \setminus V$ with covariance G_V, G_{V^c} is given by

$$\exp\left\{-\frac{1}{2}\sum_{\ell\in\mathcal{L}^*} m(\ell)\right\} \exp\left\{\sum_{e\in\mathcal{E}^*} q_e\, Z_e\right\},$$

where $\mathcal{E}^*$ denote the set of edges with one vertex in V and one in V^c, and $\mathcal{L}^*$ denotes the set of loops that intersect both V and V^c. This gives a way, first proposed by Lupu, to put the signs on the field.

- Take a Poissonian realization of the loop measure giving the local times n_x and then choose continuous local times giving t_x.
- Open each edge e that has been traversed by a loop in the soup.
- Independently, open each edge with probability $1 - \exp\{-2q_e\sqrt{t_e}\}$.
- Give each Z_x in a connected cluster the same sign using independent fair coins in each cluster.

This formulation has started with a measure on discrete time loops and added some waiting times. The original construction started with a continuous time loop measure that can be derived from the discrete time measure by adding exponential waiting times of mean one which is the same as $\chi_2^2/2$. In this formulation, one also needs to have some "trivial loops" that do not move and whose time duration have a $\chi_1^2/2$ distribution. Another nice variation due to Lupu [41] only uses a loop soup on a slightly different graph (metric graph or cable system) and some random coin flips. In this case the connected clusters for the loop soup are exactly the clusters for which one choose random coin flips.

We note that when n_x is large, it makes little difference whether we use n_x or t_x. Indeed,

$$t_x = n_x + Y_x + O(1),$$

where Y_x is an independent, mean zero, random variable with variance n_x and

$$\sqrt{2t_x} = \sqrt{2n_x} + \frac{Y_x}{\sqrt{2n_x}} + O(n_x^{-1/2}),$$

where $\{Y_x\}$ are conditionally independent given $\{n_x\}$ with mean zero and variance n_x.

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8 Continuous Gaussian field and quantum gravity

As fundamental a conformal invariant as Brownian motion in $\mathbb{C}$ is the (continuous) Gaussian free field. One way that it can be obtained is as a limit of the Gaussian field from the previous section. Let D be a simply connected bounded subdomain of $\mathbb{C}$ containing the origin. Let g_D denote the Green's function for the Laplacian: if $\mathbb{D}$ is the unit disk and $f : D \to \mathbb{D}$ is a conformal transformation with $f(z) = 0$, then $g_D(z,w) = -\log|f(w)|$. Let N be a large integer; in the spirit of nonstandard analysis [46] we can consider $N \simeq \infty$. We will use the notation from Section 3 with lattice spacing $1/N$ and corresponding sets $D_N \subset D$ and $A_N \subset \mathbb{Z}^2 \cap ND$.

Let $G_N = G_{A_N}$ denote the Green's function for the usual random walk killed upon leaving A_N, and let g_N denote the function $g_N(z,w) = (\pi/2)\, G_N(zN, wN)$. For standard $z, w \in D$ with $z \neq w$,

$$g_N(z,w) \simeq g_D(z,w)$$

(precise error estimates can be given, see, e.g., [23], but we will not discuss them). The Gaussian field on A_N, $\{Z_z : z \in A_N\}$ can be viewed as a piecewise constant function $\phi_N(z) = \sqrt{\pi/2}\, Z_{wN}$ for $z \in \mathcal{S}_w$ (we do not need to worry about the values on the boundaries of the squares $\mathcal{S}_w$). In the terminology of nonstandard analysis, the Gaussian free field ϕ on D with Dirichlet boundary conditions is the "standard part" of ϕ_N.

The macroscopic object ϕ is a little tricky because it is not defined pointwise. For example, $\phi_N(0)$ is a normal random variable of variance (up to an infinitesimal)

$$g_N(0,0) = \log N - \log|f'(0)| + c_0$$

and hence the standard part is not defined. One way to get well-defined quantities is to take averages. For example, if f is a standard L^2 function on D, then we can define $\phi(f)$ to be the standard part of

$$\int_{D_N} f(z)\, \phi_N(z)\, dA(z) = \sqrt{\frac{\pi}{2}}\, N^{-2} \sum_{z \in A_N} f(z/N)\, Z_z,$$

which is a centered normal random variable with variance

$$\int_D \int_D f(z)\, f(w)\, g_D(z,w)\, dA(z)\, dA(w).$$

This observation gives one way to construct the continuous field rigorously — as the centered Gaussian process indexed by functions f with

$$\mathbf{E}\left[\phi_f\, \phi_h\right] = \int_D \int_D f(z)\, h(w)\, g_D(z,w)\, dA(z)\, dA(w).$$

We will retain the discrete picture to consider what is sometimes called Liouville quantum gravity. This is a fancy term for the exponential $e^{\gamma\phi}$ where γ is a constant that we will choose to be nonnegative. To make sense of this, consider the field $\psi_N(z) = N^{-\gamma^2/2} \exp\{\gamma \phi_N(z)\}$. We choose this normalization so that the expected value is of order one,

$$\mathbf{E}\left[\psi_N(z)\right] = N^{-\gamma^2/2}\, \mathbf{E}\left[\exp\{\gamma\phi_N(z)\}\right] = N^{-\gamma^2/2} \exp\left\{\frac{\gamma^2\, \mathrm{Var}(\phi_N)}{2}\right\} \asymp 1.$$

We let μ_N be the random measure on D_N whose Radon-Nikodym derivative with respect to area is ψ_N, that is, $\mu_N[\mathcal{S}_z] = N^{-2}\, \psi_n(z)$. Let μ be the standard part of this measure. It is not so clear whether this make sense. By construction we can see that μ_N is a random measure on D such that $\mathbf{E}[\mu_N(D)] \asymp 1$. However, it does not follow from this calculation that the "typical" value of $\mu_N(D)$ is of order 1; it is possible that the typical value is infinitesimal. Whether or not this is true depends on the value γ.

We can do a "back of the envelope" calculation to find the critical value. Suppose Φ is a normal random variable with mean zero, variance $\sigma^2 = \log N + O(1)$, and $Y = e^{\gamma\Phi}$. If we tilt the distribution by Y, that is

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consider the random variable Φ under the measure tilted by $e^{-\gamma^2/2}\, e^{\gamma\Phi}$, the induced distribution on Φ is that of a normal random variable with mean $\gamma\sigma^2$ and variance σ^2. The original probability of getting a value as large as $\gamma\sigma^2$ is of order $e^{-\gamma\sigma^2/2}$. That is, the typical value of $\psi_N(z)$ in the tilted measure is of order $N^{\gamma^2/2}$ and the probability (in the original measure) of such of value is of order $N^{-\gamma^2/2}$. Since there are of order N^2 points, we see that critical value is $\gamma = 2$; if $\gamma < 2$, then we would expect that the measure μ would be supported on a set of $N^{2-\frac{\gamma^2}{2}}$ squares, that is, on a set of "fractal dimension" $2 - \frac{\gamma^2}{2}$.

We will now use Liouville quantum gravity to reparametrize a curve. Let us consider the loop-erased random walk which has dimension $d = 5/4$. Then we can reparametrize the curve as in (5) and get a curve whose macroscopic time duration is finite and positive. The number of points visited by a typical path is comparable to N^d and the amount of time it spends on each of these points is $N^{-d} = [\text{area}(\mathcal{S}_z)]^{-d/2}$. Suppose an independent realization of the Liouville quantum gravity is given. Then we can also reparametrize our curve so that the amount of time spent on square $\mathcal{S}_z$ is $[\sqrt{\mu_N(\mathcal{S}_z)}]^{-\alpha}$. Here we can view α as the "quantum fractal dimension" of the path chosen so that

$$\sum_z [\sqrt{\mu_N(\mathcal{S}_z)}]^{-\alpha} \asymp 1.$$

If a random set of dimension d is chosen independently of the Gaussian field, then the expected value of the left-hand side is comparable to

$$N^d\, \mathbf{E}\left[(\mu_N(\mathcal{S}_z))^{-\alpha/2}\right]$$

where z is a typical interior point for which we see that

$$\mathbf{E}\left[(\mu_N(\mathcal{S}_z))^{-\alpha/2}\right] = N^{-\alpha(1+\frac{\gamma^2}{4})}\, \mathbf{E}\left[\exp\{\alpha\gamma Z_{Nz}/2\}\right] \asymp N^{-\alpha(1+\frac{\gamma^2}{4})+\frac{\alpha^2\gamma^2}{8}}.$$

This gives the KPZ relation

$$d = \alpha\left(1 + \frac{\gamma^2}{4}\right) - \frac{\alpha^2\,\gamma^2}{8}, \qquad (11)$$

which is often written in terms of the scaling exponents x, Δ defined by $d = 2 - 2x, \alpha = 2 - 2\Delta$,

$$x = \left(1 - \frac{\gamma^2}{4}\right)\Delta + \frac{\gamma^2}{4}\,\Delta^2.$$

As in the case of the loop measure, for each $\kappa \leq 4$, there is a corresponding value of γ. In this case γ is chosen so that the quantum fractal dimension α_0 of the SLE_κ path is 1. Using (11) we can see which γ to choose for each κ.

- If $\gamma^2 = \kappa$, then the quantum fractal dimension of an independent set of Euclidean fractal dimension $1 + \frac{\kappa}{8}$ is one.

In the case of the loop-erased random walk, we choose $\gamma = \sqrt{2}$, and then we have a one-dimensional parametrization of the d-dimensional curve. For $\kappa' > 4$, a similar association is appropriate; indeed, the outer boundary of $SLE_{\kappa'}$ curves are locally like SLE_κ curves with $\kappa\,\kappa' = 16$. These values of κ, κ' share the same central charge.

One of the most exciting recent developments in conformally invariant systems has been the work of Scott Sheffield, Jason Miller, Bertrand Duplantier, and others in understanding the random geometry and surfaces produced by taking independent realizations of the Gaussian free field (and hence of the quantum gravity) and realizations of SLE_κ or $SLE_{\kappa'}$ curves and loops. I am not going to try to explain this work for two reasons: it would take too much space to give even a reasonable description and I do not feel I have sufficient expertise to do it justice. I suggest the paper [14] whose abstract starts with the inviting sentence "There is a simple way to "glue together" a coupled pair of continuum random trees (CRTs) to produce a topological sphere", but then is followed by a very technical paper of over 200 pages! Another major breakthrough by Miller and Sheffield [44] is making rigorous the relation between the $\gamma^2 = 8/3$ ($\mathbf{c} = 0$) case and combinatorial models for random graphs and the Brownian map [37].

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9 Random simple loops

A rooted self-avoiding loop (rSAL) is a path $[l_0, l_1, \ldots, l_{2n}]$ with $l_0 = l_{2n}$ and all other vertices distinct. We will call ℓ an (unrooted) self-avoiding loop (SAL) if it is an equivalence class of rooted self-avoiding loops as before. For self-avoiding loops, there are exactly $2n$ rooted loops associated to an SAL. We have retained the orientation of the loop. A self-avoiding polygon (SAP) is an equivalence class of SAL where we ignore the orientation; to each SAP of length $2n > 2$, there are 2 SALs and $4n$ rSALs.

When studying SALs or SAPs in D_N, we can either consider loops in the (scaled) lattice or the dual lattice. We note that SAPs on the dual lattice are in one-to-one correspondence with finite simply connected subsets of $\mathbb{Z}^2$ where the correspondence is given by the boundary. For finite (not necessarily simply) connected subsets we can fill in the bounded components of the complement (giving the hull of the set) and then take the outer boundary. Of course, this is not a bijection since the outer boundary of a set is the same as the outer boundary of its hull. We will be studying measures on SAPS or SALs with an emphasis on the macrosopic (that is, noninfinitesimal) diameter. Some of these measures will be infinite because they give large measure to small loops, but the measure on large loops is bounded.

We start by considering a simple to define measure on loops using the random walk measure similar to one in [24]. We will define it as a measure on SALs, but one could equally consider it as a measure on SAPs (being careful of factors of 2 since the relation between SALs and SAPs is 2-to-1). There are two variants of the measure, depending on whether the loops lie on the lattice or the dual lattice. In either case we will be considering the random walk loop measure on the original lattice. Recall that if η is a loop in the lattice, then

$$F_\eta(A) = \exp\left\{ -\sum_{\ell \subset A, \ell \cap \eta \neq \emptyset} m(\ell) \right\}.$$

Here $\ell \cap \eta \neq \emptyset$ means that the loops share a vertex. If η is a loop in the dual lattice, we define $F_\eta(A)$ in the same way but in this case $\ell \subset A$ means that the edges of ℓ are parts of boundaries of squares centered at $z \in A$, and $\ell \cap \eta \neq \emptyset$ means that ℓ includes a vertex adjacent to η. Our simple candidate for a measure is to give each η measure

$$m_A(\eta) = e^{-\beta|\eta|}\, F_\eta(A)^{-\mathbf{c}/2}, \tag{12}$$

where $\beta = \beta_c$ is a critical value and $\mathbf{c}$ denotes the central charge. Part of the conjecture is a form of *hyperscaling*, which can be stated roughly that at the critical value of β, the total measure of loops of diameter at least 1 contained in D is of order 1. The conjecture is that many of the interesting measures on loops are absolutely continuous with respect to this measure but that there may be domain corrections that will depend on the particular model studied.

One way to compensate, which will turn out to be natural at least in the case $\mathbf{c} = -2$, is to include an extra term

$$\hat{m}_A(\eta) = e^{-\beta|\eta|}\, [H_A(\eta, \partial A)\, F_\eta(A)]^{-\mathbf{c}/2},$$

where $H_A(\eta, \partial A)$ denotes an "excursion measure" term,

$$H_A(\eta, \partial A) = \sum_{x \in \eta} \mathrm{Es}_{\eta,A}(x) = \frac{1}{4} \sum_{x \in \eta} \sum_{|y-x|=1} [1 - g(y)].$$

Here $g(x) = g_{\eta,A}(x)$ is the probability that a simple random walk starting at x reaches η before A (so that $g \equiv 1$ on η), and $\mathrm{Es}_{\eta_A}(x) = -\Delta g$ is the probability that a simple random walk starting at η reaches ∂A before returning to η. If the scaled walk η is of diameter 1 and is not too close to the boundary, then $H_A(\eta, \partial A) \asymp 1$. Indeed, $(2/\pi)\, H_A(\eta, \partial A) \sim r^{-1}$ where r is chosen so that annular region between η and ∂D_A is conformally equivalent to $\{1 < |x| < e^r\}$. In particular, the continuum limit is a conformal invariant at least for transformations of the annular region. In this case one can show similarly to [16] that the limit

$$\lim_{A \uparrow \mathbb{Z}^2} H_A(\eta, \partial A)\, F_\eta(A)$$

exists and is nontrivial.

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9.1 c = 0: Self-avoiding polygons

The case $\mathbf{c} = 0$ where $\hat{m}_A(\eta)$ depends only on $|\eta|$ is a version of one of the big open questions in the intersection of probability, combinatorics, and statistical physics. The value e^β is called the connective constant and its value is not known (although it is known on the honeycomb lattice [12]). However, its continuum limit is perhaps the easiest to construct because it satisfies the restriction property: the value $\hat{m}_A(\eta)$ does not change if A changes, provided that $\eta \subset A$.

A very similar measure can be constructed from the random walk loop measure. To each unrooted loop we can associate its outer boundary. To be more precise, the set of vertices visited by an unrooted loop is a connected set and this set can become a simply connected A by filling in the finite holes. The outer boundary is the simple loop in the dual lattice given by ∂D_A. Mandelbrot [43] made the remarkable heuristic observation that the outer boundary of these loops looked like self-avoiding walks. The random walk loop measure therefore generates a measure on SAPs on the dual lattice (one could also specify or choose a random orientation to get a measure on SALs). For the continuous limit, Brownian motion, this was proved, first in [26] where it was shown that locally the paths are the same as $SLE_{8/3}$ paths. A direct construction of $SLE_{8/3}$ loops without topological constraints on a domain was done by Werner [60].

9.2 c = −2: Loop-erased loops

We will call a SAW ξ a *near-SAL* if ξ has an odd number of steps and ends distance 1 from the starting position. In other words, ξ can be turned into a SAL by adding the edge connecting the initial and terminal vertices. For each SAL η with $2n$ steps, there exist $2n$ near-SALs ξ (each with $2n-1$ steps) that produce η. For each $\xi = [x = \xi_0, \xi_1, \ldots, y = \xi_k]$ in A the quantity $4^{-k}\,F_\xi(A)$ represents the expected number of times that one views ξ if one starts a random walk at x, erases loops as they appear chronologically, and stops the walk when it leaves A. Equivalently,

$$4^{-k}\,F_\xi(A) = \sum_{\omega: x\to y, LE(\omega)=\eta} 4^{-|\omega|},$$

where the sum is over all ordinary (not necessarily self-avoiding) random walks in A from x to y whose loop-erasure is ξ. In analogy with the case of the loop measure, if we give each near-SAL measure

$$\frac{1}{4\,(|\xi|+1)}\,4^{-k}\,F_\xi(A), \tag{13}$$

then the induced measure on SALs is m_A.

Using [4], one can see that the expected number of times that the loop-erasing process starting at x (not too close to the boundary) produces a near-SAL with diameter greater than 1 is comparable to $N^{-3/4}$, and the typical number of steps of such a near-SAL is of order $N^{5/4}$. Hence the total mass of the measure in (13) for near-SALs rooted at x of diameter greater than one is comparable to N^{-2}. Summing over the $O(N^2)$ points, see that the measure μ_A of macroscopic loops is comparable to one. For macroscopic loops that are not too close to ∂A we also get that $H_{\partial A}(\eta, A)$ is comparable to one.

The measure $\hat{m}$ arises naturally in the study of uniform spanning trees. If A is a finite subset of $\mathbb{Z}^2$ with n elements, then a *wired spanning tree* is a spanning tree of the graph of $n+1$ vertices obtained by identifying all the boundary points as a single vertex we can call ∂A. Using Wilson's algorithm with ∂A as the root, we can see that the number of wired spanning trees is $4^n \det[I - Q_A] = 4^n/F(A)$, where Q_A is the matrix indexed by A with $Q_A(x,y) = 1/4$ if x, y are nearest neighbors and equals zero otherwise.

As an extension if the boundary is partitioned into two sets ∂_1 and ∂_2 and we wire ∂_1, ∂_2 separately, giving a graph of $n+2$ vertices, then the number of spanning trees is

$$\frac{4^{n+1}}{F(A)\,H_{\partial A}(\partial_1, \partial_2)},$$

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where

$$H_{\partial A}(\partial_1, \partial_2) = \sum_{x \in \partial_1} \mathrm{Es}_{\partial_1}(x) = \sum_{x \in \partial_2} \mathrm{Es}_{\partial_2}(x)$$

and $\mathrm{Es}_{\partial_j}(x)$ is the probability that a simple random walk starting at x reaches ∂_{3-j} before returning to ∂_j. Indeed, this is what is output from Wilson's algorithm if one makes ∂_1 the root and ∂_2 the initial vertex from which loop-erased random walks are chosen. In the case of an annular region with ∂_1, ∂_2 being the components of the boundary we call it a *(wired) crossing spanning tree*.

If η is an SAL of length n surrounding the origin, we say that a spanning tree includes η if all but one of the edges of η are included in the tree (it is impossible for all the edges of η to be included). We claim that the probability that a uniform spanning tree contains η is $\hat{m}(\eta)$. Indeed, if we partition the vertices on A into η, A_η^-, A_η^+ where A_η^- is the connected component of $A \setminus \eta$ containing the origin, then we can choose a tree $\mathcal{T}$ including η as follows:

- Choose any crossing spanning tree $\mathcal{T}^+$ of A_η^+ from η to ∂A in A_η and add those edges to $\mathcal{T}$.
- Choose any wired spanning tree $\mathcal{T}^-$ in A_η^-, and add those edges to $\mathcal{T}$.

Given $(\mathcal{T}^+, \mathcal{T}^-)$, $\mathcal{T}$ is determined as follows.

- There is a unique η_j such that $\mathcal{T}^+$ contains a path from η_j to ∂A. Add all the edges of η to $\mathcal{T}$ except for (η_j, η_{j+1}).

The number of choices for $\mathcal{T}^+$ is $4^{\#(A_\eta^+)+1}\,[F(A_\eta^+)\; H_{\partial A}(\eta, \partial A)]^{-1}$, and the number of choices for $\mathcal{T}^-$ is $4^{\#(A_\eta^-)}\, F(A_\eta^-)^{-1}$. Since A_η^+ and A_η^- are separate connected components and hence no loop in one intersects the other, we have

$$F_\eta(A) = \frac{F(A)}{F(A_\eta^+) F(A_\eta^+)},$$

which gives our claim. An alternative approach would be to use Wilson's algorithm to find the probability that the path from η^{j+1} to ∂A includes the path η with (η_j, η_{j+1}) removed. See [19] for another approach to these ideas.

9.3 $0 < \mathbf{c} \leq 1$: Conformal loop ensembles

Another measure on loops can be obtained from the random walk loop soup with different intensities. The construction of the Gaussian field used the soup with intensity 1/2 and we will generalize this to intensities $\mathbf{c}/2$ for $0 < \mathbf{c} \leq 1$. Given a realization of the loop soup, there are connected clusters of points. If $\mathbf{c} \leq 1/2$, these clusters will be finite. Indeed, if the clusters were not finite for $\mathbf{c} = 1/2$, then the construction of the Gaussian field as described in Section 7 would output a field with all values of the same sign. For each cluster, we can consider the outer boundary (the boundary of the unbounded component of the complement) as a SAP in the dual lattice. For each simply connected A let us write μ_A for this measure on SAPs (on the dual lattice) in A.

In order for η to be the outer boundary of a cluster two things must be true:

- The loop η is not hit by the loop soup.
- All the points "inside" η that are adjacent to η must be in the same connected component.

We call such an η an outermost loop if it also satisfies:

- There is no η' in the annular region between η and ∂D that satisfies the first two conditions and disconnects η from ∂D.

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While this measure does not have the exact form (12), we will do some heuristics to see that it is similar. First, if $A \subset A'$, we note that

$$\frac{\mu_{A'}(\eta)}{\mu_A(\eta)} = \frac{m_{A'}(\eta)}{m_A(\eta)} = \left[\frac{F_\eta(A)}{F_\eta(A')}\right]^{\mathbf{c}/2}.$$

where the right-hand side is the probability that the loop soup in A' contains a loop that intersects both η and $A' \setminus A$. Of course an outermost loop in A may no longer be an outermost loop in A'.

The continuous analogue of this construction (as well as a different construction that we will not describe here) was carried out by Sheffield and Werner [55] focusing on the outermost loops. They used the following property to characterize the measure on outermost loops. Suppose $A \subset A'$ are simply connected and we observe the outermost loops that intersect $A' \setminus A$. Let V be A with the points surrounded by these loops removed. Then the conditional distribution on the remainder of the outermost loops is that of the outermost loops of (the connected components) of V. The exact lattice construction we mention may be unsolved, but there is a closely related construction [58] that focuses only on large (macroscopic and some mesoscopic) loops in the random walk clusters and then shows that the macroscopic clusters are the same as those from Brownian clusters. It is in this regime that the coupling [34] between the random walk and Brownian loop soups works and hence they can reduce the problem to the Sheffield-Werner construction. One would expect that the exact nature of microscopic loops should not play a big factor in the scaling limit but this is still open.

Another way to get a measure on loops is to observe a field and to consider the loops that separate values of different signs. One case where this has been done is the Gaussian field. It is useful to consider an equivalent definition of the free field, this time with non-zero boundary conditions, as having the density with respect to normalized Lebesgue measure $\prod_{x \in A}(dz_x/\sqrt{2\pi})$ of

$$\sqrt{\det(I-Q)}\,\exp\left\{-\frac{1}{2}\,\mathcal{E}(\bar{z})\right\}, \quad \mathcal{E}(\bar{z}) = \frac{1}{4}\sum_e [z_x - z_y]^2,$$

where in this case the sum is over all edges $e = \{x, y\}$ with at least one vertex in A; and $z_x = 0$ for $x \in \partial A$ (other boundary conditions can be given).

Suppose a SAP η in the dual lattice is give, and let V_η^+, V_η^- denote the adjacent vertices to η that are outside and inside η respectively. We will consider the event that $z_x < c$ for $x \in V_\eta^+$ and $z_y > c$ for $y \in V_\eta^-$. We first consider the exponential term for edges that cross η. This gives a distribution on $z_x, \in V_\eta := V_\eta^+ \cup V_\eta^-$ up to an additive constant that we then fix so that average of z_x (as seen from far away) in V_η^+ is 0. We let λ be the average value in V_η^- as seen, say, from a point on the inside. (The boundary value will have local microsopic fluctuations but look constant from a macroscopic distance away.)

Given $\{z_x : x \in V\}$, we choose the rest of the Gaussian free field on the remaining points A^+, A^- to be independent fields with zero boundary condition plus a mean given by the harmonic extension of the boundary values. Away from η, this harmonic extension looks like 0 on A^+ and λ on A^-. The energy contribution given by the harmonic extension is local near η and should give a term linear in the length of η. So, roughly speaking, the probability of getting the curve η should look like

$$\frac{e^{-\beta|\eta|}\,\sqrt{F(A^+)}\,\sqrt{F(A^-)}}{\sqrt{F(A)}} = e^{-\beta|\eta|}\,F_\eta(A)^{-1/2},$$

for some β. There is a lot of hand waving here, but we can see how a form like (12) arises.

To make arguments like this rigorous in the continuum, one can reverse the operation [54, 59] One starts with a measure on loops, one finds a critical value of λ, and then given the loop one constructs independent Gaussian fields on the outside (with boundary value 0) and the inside (with boundary value λ). Then one shows that this construction combines to give a Gaussian field in the large domain. In some since the curve is a level curve for the final Gaussian field (and we can view it as a "function" of that field).

The idea of starting with a Gaussian field and defining curves and loops as a function of the field was proposed in [11] and has been developed by many under the name "imaginary geometry" to get results about SLE and loops, see, e.g., [45] A similar result for loops in the Ising model can be found in [5].

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10 SLE_κ loops

There is a direct way to define SLE_κ loops that work for all $\kappa < 8$ that is analogous to the definition for the Brownian loop measure. This defines a measure on loops in the entire plane that is invariant under dilations and rotations, but leaves open the question how to modify the measure for a bounded domain.

Using the Loewner equation with a driving function of a killed process in a quasi-invariant distribuution, one can define a σ-finite measure on loops rooted at a particular point. We write $\gamma_t = \gamma[0,t]$ and D_t for the unbounded component of $\mathbb{C} \setminus \gamma_t$. For the moment, we parametrize the curves by capacity in the upper half plane: if $F : \mathbb{C} \setminus \mathbb{D} \to D_t$ is a conformal transformation fixing ∞, then as $z \to \infty$, $|F'(z)| \sim e^t\,|z|$.

- The set of loops with total capacity greater than t is $c\,e^{-t(d-2)}$ for some fixed constant c.
- Conditioned on the total capacity of the loop being greater than t, the conditional distribution of $\gamma(s), s \geq t$ given γ_t is that of chordal SLE_κ from $\gamma(t)$ to $\gamma(0)$ in D_t.

If $f_r(z) = rz$ denotes dilation by r then the measure μ_0 satisfies $f_r \circ \mu_0 = r^{2-d}\,\mu_0$. We can also consider this as a measure on curves with the natural parametrization. Let $T_\gamma = \mathrm{Cont}_d(\gamma)$. Then (by choosing c appropriately) we get a measure on naturally parametrized loops with (recall that a loop of capacity t typically has content of order t^d)

- The set of loops with $T_\gamma \geq T$ has measure $T^{1-\frac{d}{2}}$.

As in the case of Brownian loops, we will try to integrate the rooted loop measure over the starting points to give a measure on unrooted loops. As before, this leads to overcounting so we compensate by considering the measure ν_z given by

$$\frac{d\nu_z}{d\mu_z} = \frac{1}{T_\gamma}, \qquad \nu_z = \int_{\mathbb{C}} \nu_z\, dA(z).$$

Again, we think of this as a measure on unrooted loops. Even for the measure on rooted loops, we get the scaling relation $f_r \circ \nu = \nu$.

Laurie Field and I were studying this and had gotten this far; indeed, one of the motivations for understanding natural parametrization was to try a construction like this in order to give a measure on loops of the type suggested by Kontsevich and Suhov [22]. However, there was a technical question that we were unable to answer that was necessary to continue this program. There is a property of Brownian loops that it almost "obvious" and is used in the proof of the conformal invariance: if $\gamma(t), 0 \leq t \leq 1$ has the distribution of a Brownian bridge and $0 < s < 1$, then the distribution of $\tilde{\gamma}(t) := \gamma(t+s) - \gamma(s)$ is also that of a Brownian bridge. (Here addition is modulo 1.) The analogue for the SLE_κ loop measure is that the measure conditioned on fixed Minkowski content has the same property. This has recently been proved by Zhan [63].

The conformal invariance here is only for the dilations. There is still the hard question about how to restrict the measure to bounded domains. This does not arise for the Brownian loop measure because it satisfies the restriction property. There is not a unique possibility, and the exact version should depend on the particular problem being analyzed.

11 Scaling limit of loop-erased walk

Suppose z, w are distinct boundary points on D. We will consider two processes:

- Chordal SLE_2 γ from z' to w' in D.
- Take the discrete approximation D_N and corresponding boundary points $z, w \in D_N$, and let η be a (scaled) LERW from z to w

These processes are close and, in this section we discuss recent results showing that the "naturally parametrized" curves are close. At the moment this is the only process for which this strong convergence is known.

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To establish the result, we start by proving a result about the LERW that can be considered a "local limit theorem". We compare the probability that the LERW goes through the origin with the probability that a chordal SLE_2 path goes through the square $\mathcal{S}_0$ of side length N^{-2}. Let $r_D = r_D(0)$ and $S_D = S_D(0; z, w)$ be the parameters as in Section 6. Using stochastic calculus techniques one can show that there exist c_0, u (independent of D, z, w) such that

$$\mathbf{P}\{\gamma \cap \mathcal{S} \neq \emptyset\} = c_0 \, N^{-5/4} \, r_D^{-5/4} \, S_D^3 \, [1 + O(N^{-u})].$$

(This was established for a disk rather than a square in [31], but the argument can be adapted for a square. The constant c_0, which is different for squares and disks, is not known explicitly.)

We will describe work in [4, 28] that established the analogous result for LERW: there exists an absolute c_1 such that for all domains

$$\mathbf{P}\{0 \in \eta\} = c_1 \, N^{-5/4} \, r_D^{-5/4} \, S_D^3 \, [1 + O(N^{-u})].$$

Note that this not only gives the correct scaling exponent (which had been established by Kenyon [20]) but also the dependence of the constant factor in the asymptotics to the domain, establishing that it is a conformally covariant quantity. The proof combines a key ingredient of Kenyon's proof with the machinery of loop measures, this time with measures that can take negative values.

Let A be a finite, simply connected subset of $\mathbb{Z}^2$ containing the origin, and let D_A be the corresponding "union of squares" domain. We will not scale D_A, so if $f : \mathbb{D} \to A$ is a conformal transformation with $f(0) = 0$, then $r_A := |f'_A(0)| \asymp \mathrm{dist}(0, \partial A)$. if $z_1, z_2 \in \partial A$, we can also find the angle S_A from this map. We will compare three numbers: $H_A(z, w)$, the measure of usual random walks from z to w in A; $\hat{H}_A(z, w; \vec{01})$ the measure of such walks whose loop erasure uses the directed edge $\vec{01}$; and $\hat{H}_A(z, w; \vec{10})$, the measure of such walks that use the edge $\vec{10}$. The probability that the undirected edge $\{0, 1\}$ is used is then

$$\frac{\hat{H}_A(z, w; \vec{01}) + \hat{H}_A(z, w; \vec{10})}{H_A(z, w)}.$$

Using a determinantal formula first given by Fomin [17], one can give an exact expression for the difference,

$$\hat{H}_A(z, w; \vec{01}) - \hat{H}_A(z, w; \vec{10}) = \frac{1}{4} \, F_{01}(A) \, [H_{A'}(z, 0) \, H_{A'}(w, 1) - H_{A'}(z, 1) \, H_{A'}(w, 0)] \, ,$$

where $A' = A \setminus \{0, 1\}$. Unfortunately, this is a formula for a difference rather than a sum on the left-hand side. Kenyon's trick is to change some of the weights to negative; more precisely, we can draw a vertical half line ("zipper") on the edges of the dual graph $\{\frac{1}{2} + iy : -y_0 < y < 0\}$ where $\frac{1}{2} - iy_0$ is the first point on ∂D_A reached. Then for each edge of A that crosses the zipper we give weight $-1/4$ rather than $1/4$. This new assignment of edge weights gives a new measure on paths, and hence loops, that we will denote as q. Fomin's identity is a combinatorial bijection that works with any weights on the bonds; in particular,

$$\hat{H}^q_A(z, w; \vec{01}) - \hat{H}^q_A(z, w; \vec{10}) = \frac{1}{4} \, F^q_{01}(A) \, [H^q_{A'}(z, 0) \, H^q_{A'}(w, 1) - H^q_{A'}(z, 1) \, H^q_{A'}(w, 0)] \, ,$$

where we use the superscript q to mean quantities computed with that measure.

We use expressions as before

$$\hat{H}_A(z, w; \vec{01}) = \sum_\eta 4^{-|\eta|} \, F_\eta(A), \quad \hat{H}^q_A(z, w; \vec{01}) = \sum_\eta (-1)^{J(\eta)} \, 4^{-|\eta|} \, F^q_\eta(A),$$

where the sum is over all SAWs η from z to w using the directed edge $\vec{01}$. Here $J(\eta)$ is the number of times that η crosses the zipper. We now use simple connectivity of the domain and some simple topology to observe two facts:

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- If z, w are ordered correctly, every SAW from z, w that uses the directed edge $\overrightarrow{01}$ crosses the zipper an even number of times while SAWs that use $\overrightarrow{10}$ cross an odd number of times.

- Any loop that crosses the zipper an odd number of times must intersect every η using $\overrightarrow{01}$ or $\overrightarrow{10}$.

This gives

$$\hat{H}^q_A(z,w;\overrightarrow{01}) - \hat{H}^q_A(z,w;\overrightarrow{10}) = \exp\{-2m(O_A)\} \left[\hat{H}_A(z,w;\overrightarrow{01}) + \hat{H}_A(z,w;\overrightarrow{10})\right],$$

where O_A denotes the set of loops that intersect the zipper an odd number of times. This gives an exact expression for the quantity we want in terms of random walk quantities (including some for the signed measure q):

$$\frac{1}{4}\, F^q_{01}(A)\, e^{2m(O_A)} \left[\frac{H^q_{A'}(z,0)}{H_A(0,z)}\, \frac{H^q_{A'}(w,1)}{H_A(0,w)} - \frac{H^q_{A'}(z,1)}{H_A(0,z)}\, \frac{H^q_{A'}(w,0)}{H_A(0,w)}\right] \left[\frac{H_A(z,w)}{H_A(0,z)\, H_A(0,w)}\right]^{-1}.$$

There is a lot of machinery to handle random walk convergence to Brownian motion and in two dimensions one can often get good estimates uniform over all boundary conditions. There is work involved for sure, but we show that

$$\frac{H^q_{A'}(z,0)}{H_A(0,z)}\, \frac{H^q_{A'}(w,1)}{H_A(0,w)} - \frac{H^q_{A'}(z,1)}{H_A(0,z)}\, \frac{H^q_{A'}(w,0)}{H_A(0,w)} = c_1\, r_A^{-1}\, [S_A + O(r_A^{-u})],$$

$$\left[\frac{H_A(z,w)}{H_A(0,z)\, H_A(0,w)}\right]^{-1} = c_2\, [S_A^2 + O(r_A^{-u})],$$

and it is not hard to show that $F^q_{01}(A) = c_3 + O(r_A^{-u})$. The final estimate boils down to

$$m[O(A)] = \frac{1}{8} \log r_A + c_4 + o(r_A^{-u}).$$

This requires comparison to the Brownian loop measure. Suppose A_n is the discrete ball of radius e^n. Then $m[O(A_{n+1})] - m[O(A_n)]$ denotes the measure of loops in A_{n+1} that are not contained in A_n and intersect the zipper an odd number of times. For n large, this boils down to estimating the measure of loops of odd winding number about the origin, and by the strong coupling of random walk and Brownian loop measures, this is about the same as the Brownian motion loop measure of loops in the disk of e^{n+1} that are not in disk of e^n and have odd winding number about the origin. By conformal invariance, this is independent of n and a computation using Brownian bubbles as in (6) gives the value 1/8. Being more careful about the approximation, we get

$$m[O(A_{n+1})] - m[O(A_n)] = \frac{1}{8} + O(e^{-un}).$$

More general domains than disks are handled similarly, again using the coupling and the conformal invariance of the Brownian loop measure.

Given the sharp estimate we can establish the strong scaling limit for LERW. Let us consider our domain D with two boundary points and let us view the scaled LERW at a macroscopic scale. It was shown in [32] that if we ignore parametrization, the path of the LERW looks like a chordal SLE_2. In [27] it is shown how to combine these ideas with the sharp estimate for LERW above to show that the scaled natural parametrization of the LERW also converges to (an absolute constant times) the Minkowski content of the SLE path. While the proof is technical, the basic idea is as follows. Suppose we have seen part of the curve. Then the expected total length of a curve given the initial condition is the length of that segment plus the expected length of the remaining curve, see (10). A similar (and more elementary) formula holds for the number of steps of the LERW. The expected length of the remaining curve given the curve is given by the integral of the Green's function (discrete or continuous). Using the estimate in [4] (and the fact that the estimate does not require smoothness on the boundaries), the two expected lengths are the same. Roughly speaking, the difference of the lengths in the coupling is a martingale whose quadratic variation is very small and hence must be small.

For ICM 2018 participants only

While the structure of the proof in [27] is potentially applicable to other models, it requires the very sharp estimates for the discrete model. At the moment, there is no other model for which the Green's function can be estimated so precisely. A similar, but at the moment not sufficiently precise, result about the Ising model was shown in [9]; the technique of negative weights above is related to the spinors in that paper. The other model for which there is a relatively strong local theorem is the percolation exploration process, see [18].

References

[1] Vincent Beffara. The dimension of the SLE curves. *Ann. Probab.*, 36(4):1421–1452, 2008.

[2] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuclear Phys. B*, 241(2):333–380, 1984.

[3] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov. Infinite conformal symmetry of critical fluctuations in two dimensions. *J. Statist. Phys.*, 34(5-6):763–774, 1984.

[4] Christian Beneˇs, Gregory F. Lawler, and Fredrik Viklund. Scaling limit of the loop-erased random walk Green's function. *Probab. Theory Related Fields*, 166(1-2):271–319, 2016.

[5] Stéphane Benoist and Clément Hongler. The scaling limit of critical ising interfaces is cle (3). 2016.

[6] David Brydges, Jürg Fröhlich, and Thomas Spencer. The random walk representation of classical spin systems and correlation inequalities. *Comm. Math. Phys.*, 83(1):123–150, 1982.

[7] John Cardy. *Scaling and renormalization in statistical physics*, volume 5 of *Cambridge Lecture Notes in Physics*. Cambridge University Press, Cambridge, 1996.

[8] John L. Cardy. Critical percolation in finite geometries. *J. Phys. A*, 25(4):L201–L206, 1992.

[9] Dmitry Chelkak, Clément Hongler, and Konstantin Izyurov. Conformal invariance of spin correlations in the planar Ising model. *Ann. of Math. (2)*, 181(3):1087–1138, 2015.

[10] Philippe Di Francesco, Pierre Mathieu, and David Sénéchal. *Conformal field theory*. Graduate Texts in Contemporary Physics. Springer-Verlag, New York, 1997.

[11] Julien Dubédat. SLE and the free field: partition functions and couplings. *J. Amer. Math. Soc.*, 22(4):995–1054, 2009.

[12] Hugo Duminil-Copin and Stanislav Smirnov. The connective constant of the honeycomb lattice equals $\sqrt{2+\sqrt{2}}$. *Ann. of Math. (2)*, 175(3):1653–1665, 2012.

[13] Bertrand Duplantier. Liouville quantum gravity, KPZ and Schramm-Loewner evolution. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pages 1035–1061. Kyung Moon Sa, Seoul, 2014.

[14] Bertrand Duplantier. Liouville quantum gravity, KPZ and Schramm-Loewner evolution. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. III*, pages 1035–1061. Kyung Moon Sa, Seoul, 2014.

[15] E. B. Dynkin. Gaussian and non-Gaussian random fields associated with Markov processes. *J. Funct. Anal.*, 55(3):344–376, 1984.

[16] Laurence S. Field and Gregory F. Lawler. Reversed radial SLE and the Brownian loop measure. *J. Stat. Phys.*, 150(6):1030–1062, 2013.

For ICM 2018 participants only

[17] Sergey Fomin. Loop-erased walks and total positivity. *Trans. Amer. Math. Soc.*, 353(9):3563–3583, 2001.

[18] Christophe Garban, Gábor Pete, and Oded Schramm. Pivotal, cluster, and interface measures for critical planar percolation. *J. Amer. Math. Soc.*, 26(4):939–1024, 2013.

[19] Adrien Kassel and Richard Kenyon. Random curves on surfaces induced from the Laplacian determinant. *Ann. Probab.*, 45(2):932–964, 2017.

[20] Richard Kenyon. The asymptotic determinant of the discrete Laplacian. *Acta Math.*, 185(2):239–286, 2000.

[21] Harry Kesten. The work of Stanislav Smirnov. In *Proceedings of the International Congress of Mathematicians. Volume I*, pages 73–84. Hindustan Book Agency, New Delhi, 2010.

[22] M. Kontsevich and Y. Suhov. On Malliavin measures, SLE, and CFT. *Tr. Mat. Inst. Steklova*, 258(Anal. i Osob. Ch. 1):107–153, 2007.

[23] Michael J. Kozdron and Gregory F. Lawler. Estimates of random walk exit probabilities and application to loop-erased random walk. *Electron. J. Probab.*, 10:1442–1467, 2005.

[24] Michael J. Kozdron and Gregory F. Lawler. The configurational measure on mutually avoiding SLE paths. In *Universality and renormalization*, volume 50 of *Fields Inst. Commun.*, pages 199–224. Amer. Math. Soc., Providence, RI, 2007.

[25] G. Lawler. Conformal invariance, universality, and the dimension of the Brownian frontier. In *Proceedings of the International Congress of Mathematicians, Vol. III (Beijing, 2002)*, pages 63–72. Higher Ed. Press, Beijing, 2002.

[26] Gregory Lawler, Oded Schramm, and Wendelin Werner. Conformal restriction: the chordal case. *J. Amer. Math. Soc.*, 16(4):917–955, 2003.

[27] Gregory F. Lawler. Loop-erased random walk. In *Perplexing problems in probability*, volume 44 of *Progr. Probab.*, pages 197–217. Birkhäuser Boston, Boston, MA, 1999.

[28] Gregory F. Lawler. The probability that planar loop-erased random walk uses a given edge. *Electron. Commun. Probab.*, 19:no. 51, 13, 2014.

[29] Gregory F Lawler. Topics in loop measures and the loop-erased walk. 2017.

[30] Gregory F. Lawler and Vlada Limic. *Random walk: a modern introduction*, volume 123 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2010.

[31] Gregory F. Lawler and Mohammad A. Rezaei. Minkowski content and natural parameterization for the Schramm-Loewner evolution. *Ann. Probab.*, 43(3):1082–1120, 2015.

[32] Gregory F. Lawler, Oded Schramm, and Wendelin Werner. Conformal invariance of planar loop-erased random walks and uniform spanning trees. *Ann. Probab.*, 32(1B):939–995, 2004.

[33] Gregory F. Lawler and Scott Sheffield. A natural parametrization for the Schramm-Loewner evolution. *Ann. Probab.*, 39(5):1896–1937, 2011.

[34] Gregory F. Lawler and José A. Trujillo Ferreras. Random walk loop soup. *Trans. Amer. Math. Soc.*, 359(2):767–787, 2007.

[35] Gregory F. Lawler and Wendelin Werner. The Brownian loop soup. *Probab. Theory Related Fields*, 128(4):565–588, 2004.

For ICM 2018 participants only

[36] Gregory F. Lawler and Wang Zhou. *SLE* curves and natural parametrization. *Ann. Probab.*, 41(3A):1556–1584, 2013.

[37] Jean-François Le Gall. Random geometry on the sphere. In *Proceedings of the International Congress of Mathematicians—Seoul 2014. Vol. 1*, pages 421–442. Kyung Moon Sa, Seoul, 2014.

[38] Yves Le Jan. Markov loops and renormalization. *Ann. Probab.*, 38(3):1280–1319, 2010.

[39] Yves Le Jan. *Markov paths, loops and fields*, volume 2026 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 38th Probability Summer School held in Saint-Flour, 2008, École d'Été de Probabilités de Saint-Flour. [Saint-Flour Probability Summer School].

[40] Paul Lévy. Le mouvement brownien plan. *Amer. J. Math.*, 62:487–550, 1940.

[41] Titus Lupu. From loop clusters and random interlacements to the free field. *Ann. Probab.*, 44(3):2117–2146, 2016.

[42] Titus Lupu and Wendelin Werner. A note on Ising random currents, Ising-FK, loop-soups and the Gaussian free field. *Electron. Commun. Probab.*, 21:Paper No. 13, 7, 2016.

[43] Benoit B. Mandelbrot. *The fractal geometry of nature*. W. H. Freeman and Co., San Francisco, Calif., 1982. Schriftenreihe für den Referenten. [Series for the Referee].

[44] Jason Miller and Scott Sheffield. Liouville quantum gravity and the brownian map i: The qle(8/3,0) metric.

[45] Jason Miller and Scott Sheffield. Imaginary geometry III: reversibility of SLE_κ for $\kappa \in (4,8)$. *Ann. of Math. (2)*, 184(2):455–486, 2016.

[46] Edward Nelson. *Radically elementary probability theory*, volume 117 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1987.

[47] Charles M. Newman. The work of Wendelin Werner. In *International Congress of Mathematicians. Vol. I*, pages 88–95. Eur. Math. Soc., Zürich, 2007.

[48] Bernard Nienhuis. Exact critical point and critical exponents of $\mathrm{O}(n)$ models in two dimensions. *Phys. Rev. Lett.*, 49(15):1062–1065, 1982.

[49] Bernard Nienhuis. Critical behavior of two-dimensional spin models and charge asymmetry in the Coulomb gas. *J. Statist. Phys.*, 34(5-6):731–761, 1984.

[50] Steffen Rohde and Oded Schramm. Basic properties of SLE. *Ann. of Math. (2)*, 161(2):883–924, 2005.

[51] H. Saleur and B. Duplantier. Exact determination of the percolation hull exponent in two dimensions. *Phys. Rev. Lett.*, 58(22):2325–2328, 1987.

[52] Oded Schramm. Scaling limits of loop-erased random walks and uniform spanning trees. *Israel J. Math.*, 118:221–288, 2000.

[53] Oded Schramm. Conformally invariant scaling limits: an overview and a collection of problems. In *International Congress of Mathematicians. Vol. I*, pages 513–543. Eur. Math. Soc., Zürich, 2007.

[54] Oded Schramm and Scott Sheffield. A contour line of the continuum Gaussian free field. *Probab. Theory Related Fields*, 157(1-2):47–80, 2013.

[55] Scott Sheffield and Wendelin Werner. Conformal loop ensembles: the Markovian characterization and the loop-soup construction. *Ann. of Math. (2)*, 176(3):1827–1917, 2012.

For ICM 2018 participants only

[56] Stanislav Smirnov. Towards conformal invariance of 2D lattice models. In *International Congress of Mathematicians. Vol. II*, pages 1421–1451. Eur. Math. Soc., Zürich, 2006.

[57] K. Symanzik. Euclidean quantum field theory. In *Scuoloa internaionale di Fisica "Enrico Fermi", XLV Coso*, pages 152–223. Academic Press, 1969.

[58] Tim van de Brug, Federico Camia, and Marcin Lis. Random walk loop soups and conformal loop ensembles. *Probab. Theory Related Fields*, 166(1-2):553–584, 2016.

[59] Menglu Wang and Hao Wu. Level lines of Gaussian free field I: Zero-boundary GFF. *Stochastic Process. Appl.*, 127(4):1045–1124, 2017.

[60] Wendelin Werner. The conformally invariant measure on self-avoiding loops. *J. Amer. Math. Soc.*, 21(1):137–169, 2008.

[61] David Bruce Wilson. Generating random spanning trees more quickly than the cover time. In *Proceedings of the Twenty-eighth Annual ACM Symposium on the Theory of Computing (Philadelphia, PA, 1996)*, pages 296–303. ACM, New York, 1996.

[62] Zhen-Hang Yang. Hölder, Chebyshev and Minkowski type inequalities for Stolarsky means. *Int. J. Math. Anal. (Ruse)*, 4(33-36):1687–1696, 2010.

[63] Dapeng Zhan. Sle loop measures.

[64] Dapeng Zhan. Reversibility of chordal SLE. *Ann. Probab.*, 36(4):1472–1494, 2008.

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