Random walks, boundaries and measures in Conformal Dynamical System

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Fatou and Julia

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Theorem (No Julia set)

If $f \colon S \to S$ is conformal on a hyperbolic surface S, then the Fatou set is the whole surface $F(f) = S$.

Both the Fatou set and the Julia set are forward and backward f-invariant. Dynamics restricted to Fatou components can be classified into 4 types (attracting, parabolic, and 2 types of irrational rotations).

Theorem (Sulllivan's no wandering domain theorem)

Every component of the Fatou set eventually cycles, and there are finitely many periodic components.

polynomial dynamics

Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a polynomial and $\mathcal{A}(\infty)$ be the superattracting basin of ∞. Its complement $K(f) = \widehat{\mathbb{C}} \setminus \mathcal{A}(\infty)$ is called the filled Julia set. By Böttcher's theorem (superattracting basin either is conformally conjugate to z^d or contains another critical point), there is a dichotomy

Theorem

The filled Julia set $K(f)$ is connected iff its complement $\mathcal{A}(\infty)$ is conformally conjugate to the action of z^d on the unit disk $\mathbb{D}$.

polynomial dynamics

The closure of each (super)attracting basin contains the Julia set. Hence $J(f) = \partial \mathcal{A}(\infty) = \partial K(f)$.

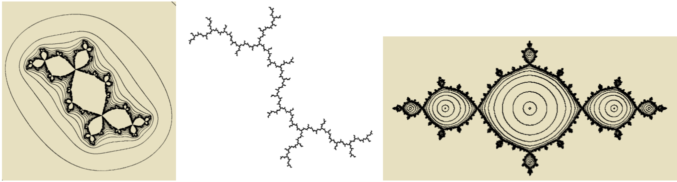

Figure: Connected Julia sets of polynomials $z^2 + (-0.1226 + 0.7449\mathbf{i})$, $z^2 + i$, and $z^2 - 1$.

harmonic measures for polynomials

We moreover assume that $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a *hyperbolic* polynomial with a connected Julia set $J = J(f)$. The basin of ∞ is also denoted by $\Omega = \mathcal{A}(\infty)$.
Hyperbolic: equipped with some conformal metric, $|f'(z)| > 1$ for all $z \in J(f)$. That is, f is expanding on the Julia set J.

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Definition

The harmonic measure $\{\nu_x\}_{x\in\Omega}$ is a family of Borel probabilities $\{\nu_x\} \subseteq \mathcal{B}(J)$ such that the following (Solution of Dirichlet problem) holds: for all continuous function $\phi\colon J \to \mathbb{R}$, the function

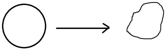
$$\widetilde{\phi}(x) := \int_{z\in J} \phi(z) d\nu_x(z), x \in \Omega, \tag{1.1}$$

is a harmonic extension of ϕ.

harmonic measures for polynomials

Consider a Riemann mapping $\phi \colon \mathbb{D} \to \Omega$ from the unit disk to the basin of ∞. The harmonic measure for $\mathbb{D}$ (seen from 0) is the Lebesgue measure $\nu_{0,\mathbb{D}} = \lambda$.

Recall: harmonicity is preserved by conformal maps. If ϕ extends continuously to $\partial\mathbb{D}$, then $\nu_{\phi(0),\Omega} = \phi_*\lambda$. It remains true in general by Fatou's theorem (angular limit of ϕ exists λ-a.e.)



ergodic properties of the harmonic measure: Brolin and Lyubich

The harmonic measure $\nu = \nu_\infty$ seen from ∞ is f-invariant and supported on the Julia set J. (By Böttcher's theorem, choose ϕ such that $f\phi(z) = \phi(z^d)$)

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The harmonic measure ν is the measure of maximal entropy.

Recall: variational principle for entropy:

$$h_\mu(f) \leqslant h_{\mathrm{top}}(f) = \ln(\deg f). \tag{1.2}$$

Motivation of our work: generalize the classical harmonic measure.

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Sullivan's dictionary

Discrete (finitely generated) (torsion-free) subgroup G of $\mathrm{Isom}_+(\mathbb{H}^3) = \mathsf{Conf}(\widehat{\mathbb{C}}) = PSL(2,\mathbb{C})$ (acting on $S^2 = \widehat{\mathbb{C}} = \partial\mathbb{H}^3$).

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properties	Kleinian groups	rational maps
dichotomy: chaotic v.s. normal	limit set v.s. ordinary set	Julia set v.s. Fatou set
parameters	generators	coefficients(preimages)
density of expansion in chaotic part	dense in limit set hyperbolic fixed points	dense in Julia set expanding periodic points
finiteness of normal part	Ahlfors finiteness theorem	no wandering domain
hyperbolicity	convex cocompact	expansion on Julia set
structural stability of hyperbolicity	True	True
geometrization	Cannon's conjecture	Thurston's characterization
	...	...

Generalization of harmonic measures?

The simplest case: hyperbolic rational maps. Restricting to the Julia set, we obtain an expanding map $f \colon J \to J$.

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What we have done: build a discrete harmonic measure given by random walk on a self-similar graph associated with the expanding dynamics.

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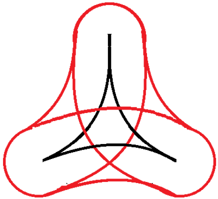
For each geodesic triangle, each edge is contained in the δ-neighborhood of other 2 edges.



Boundaries

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The *Gromov boundary* is

$$\partial X = \left\{ \{x_n\}_{n \in \mathbb{Z}_+} : \lim_{n,m \to \infty} \langle x_n, x_m \rangle_o = \infty \right\} / \sim,$$

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The *horofunction boundary* is the boundary of the embedding image of $y \mapsto \beta(\cdot, y)$ w.r.t. pointwise convergence topology.

Random walk and Martin boundary

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Theorem (Poisson-Martin representation)

For every positive harmonic function h on X, there is a positive Borel measure ν_h on $\partial_M X$ such that

$$h(x) = \int_{\partial_M X} K(x,\xi) \mathrm{d}\nu_h(\xi). \tag{2.1}$$

Relations between the boundaries

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Theorem (A. Ancona's Inequality, 1988)

$\forall \delta \geqslant 0, \exists C = C(\delta)$, *for* $x, y \in X$, $z \in N_\delta([x,y])$,

$$F(x,z)F(z,y) \leqslant F(x,y) \leqslant C(\delta)F(x,z)F(z,y).$$

Conformal measure

Patterson-Sullivan measure

$$\mu_s = \lim_{n\to\infty} \left(\sum_{|g|\leqslant n} e^{-s|g|} \right)^{-1} \left(\sum_{|g|\leqslant n} e^{-s|g|} \delta_g \right) .$$

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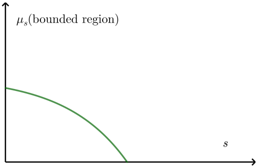

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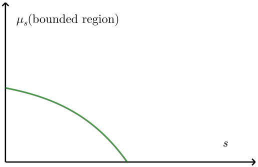

v - volume growth rate $\lim_{n\to\infty} \log(\#B(0,n)/n)$.

$$0 < s < v \implies \sum_{|g|\leqslant n} e^{-s|g|} \asymp \sum_{k=0}^{n} e^{(v-s)k} \to 1/(1-e^{v-s}).$$

$s \to v \implies \mu_v$ is supported on boundary (divergence type)

Harmonic measure

$\mu \in \mathcal{M}(\Gamma)$ - transition probability.
$\mu^{(n)}$ - n-th iteration of μ.
Fact: μ is transient $\implies$ $\mu^{(n)}$ "converges" to a boundary distribution μ_h.

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Theorem (characterization of Poisson boundary)

*If h is a **bounded harmonic** function on X, then there exists $\varphi \in L^\infty(\partial X, \mu_h)$, such that*

$$h(x) = \int \varphi K(x, \cdot) d\mu_h.$$

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$l_G = \lim_{n\to\infty} \mathbb{E}(d_G(1, Z_n(g)))/n, l = \lim_{n\to\infty} \mathbb{E}(d(1, Z_n(g)))/n$ - *drift*

Theorem (S. Blachère, P. Haïssinsky , P. Mathieu, 2009)

The following are equivalent:

- *The equality of $l_G \leqslant vl$ holds.*
- $\mu_v \asymp \mu_h$.
- μ_h *is quasi-conformal.*

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Distance expanding dynamical systems

Definition. $f\colon X \to X$ is called λ-distance expanding if

$$\exists \xi > 0 \forall d(x,y) < \xi, d(fx, fy) \geqslant \lambda d(x,y).$$

$S = \{R_1, \cdots, R_n\}$ - Markov partition

- $\operatorname{int} R_i \cap \operatorname{int} R_j = \emptyset$ if $i \neq j$;
- $R_i = \overline{\operatorname{int} R_i}$;
- $f(\operatorname{int} R_i) \cap \operatorname{int} R_j \neq \implies f(\operatorname{int} R_i) \supset \operatorname{int} R_j$.

$A_{R_i R_j} = 1 \iff f(\operatorname{int} R_i) \supset \operatorname{int} R_j$.
semi-conjugacy
$\left(\Sigma_A^+ = \{(u_n)_{n\geqslant 0} \in S^{\mathbb{Z}_{\geqslant 0}} \colon A_{u_i u_{i+1}} = 1\}, \sigma_A\right) \to (X, f)$.

Tile Graph

Vertices:
$\Gamma^0 = S^\omega = \{u_0 \cdots u_n := u_0 \cap \cdots \cap f^{-n} u_n : A_{u_i u_{i+1}} = 1\} \cup \{\emptyset\}$.
(called *tiles*)
Edges: $u - v$ iff $\big||u| - |v|\big| \leqslant 1$ and $u \cap v \neq \emptyset$.
d is called a *visual metric* if for some $\Lambda > 1$,

- $\mathrm{dist}(x, y) \gtrsim \Lambda^{-n}$, where x, y are disjoint n-tiles (in the sense of closed subsets);
- $\mathrm{diam}(x) \asymp \Lambda^{-|x|}$;

Fact. Γ is Gromov hyperbolic with Gromov boundary X, and the visual metric is Hölder equivalent to d.

Tile graph

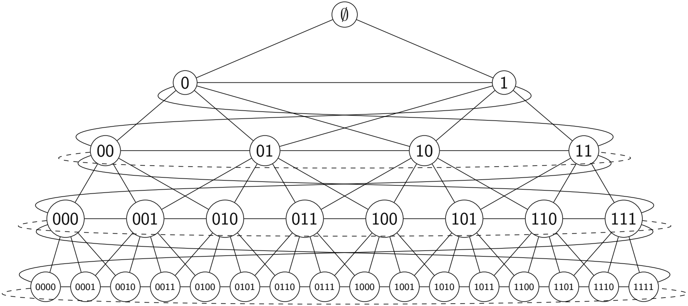

Figure: The tile graph of the doubling map on the circle.

Adapt the random walk to dynamics

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We moreover assume that:
The random walk always goes down: for all $x, y \in \Gamma$, if $p(x, y) > 0$, then $|y| > |x|$.
The random walk inherits the edges of the graph: for all $x, y \in \Gamma$, if $|y| = |x| + 1$ and $d(x, y) = 1$, then $p(x, y) > 0$.

Our results

The Martin boundary of the random walk on the tile graph maps surjectively to the phase space:

Theorem

Suppose that $f\colon X \to X$ is an open transitive distance-expanding map on a compact metric space (X, ρ), α is a sufficiently fine Markov partition, and (Γ, p) is random walk on the tile graph $\Gamma = \Gamma(f, \alpha)$ with p satisfying the assumptions in the previous page. Then there is a natural surjection Φ from the Martin boundary $\partial_M\Gamma$ of (Γ, p) to X.

Our results

Is the Martin boundary homeomorphic to the phase space?
Not true. We provide a counterexample. Even for the z^2 map on the unit circle, it is not true when the transition probability p is unbalanced. The radial growth rate of the Green function at a single point $x \in X$ may be different.
However, if the random walk is irreducible, the result of A. Ancona implies the homeomorphism of the Martin boundary and the phase space.

Our results

Based on the asymptotic quantities including

$$l_G := \lim_{n\to+\infty} -n^{-1}\log(G(Z_0, Z_n)) \text{ and } l := \lim_{n\to+\infty} -n^{-1}|Z_n|, \quad (2.2)$$

we can give a formula of the fractal dimension of the harmonic measure.

Theorem

Under the notations and hypotheses above, if X is equipped with an a-visual metric ρ for a sufficiently small constant $a > 0$, then the packing dimension of the harmonic measure ν on X is equal to $\dim_P \nu = \frac{l_G}{al}$.

The packing dimension of a measure is:

$$\dim_P \mu = \inf\{\dim_P(A) : A \subseteq X, \mu(A) > 0\} = \inf\{\dim_P(A) : A \subseteq X, \mu(A) =$$

The packing dimension is equal to the supremum of the pointwise local dimension.

Conformal measure v.s. harmonic measure

The harmonic measure ν is f-quasi-invariant. If we moreover assume that $p(x, y) > 0$ implies $|y| = |x| + 1$, then ν is f-invariant.
Conformality $f^* \mu = d\mu$ ($d =$ number of preimages)
quasi-conformality $f^* \mu_c \asymp d\mu_c$.
Questions:

- Conformal measure/measure of maximal entropy $\asymp$ / $\perp$ harmonic measure?
- Is the Hausdorff dimension of the harmonic measure the same?
- Is the fundamental inequality still true? Variation principle understanding?
- Can we apply the thermodynamical formalism and realize the harmonic measure as a measure of maximal pressure? what is the potential?